Today

Experts/Multiplicative Weights.

Experts/Zero-Sum Games Equilibrium.

Boosting and Experts.

Routing and Experts.

The multiplicative weights framework.

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

How well do you do?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

Adversary designs setup to watch who you choose, and make that expert make a mistake.

$n-1$!

Concept Alert.

Note.

Adversary:
makes you want to look bad.
"You could have done so well...
but you didn't! ha..ha... ha.

Technical Term: Regret.

Analysis of Algorithms: do as well as possible!
Minimize Regret $\equiv$ Loss.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

What you would do anyway!

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a mistake,
|"perfect" experts| drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
⋮
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert $\rightarrow$ at most $\log n$ mistakes!

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?
each incorrect expert weight multiplied by $\frac{1}{2}$!
total weight decreases by
factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
mistake $\rightarrow \geq$ half weight with incorrect experts
($\geq \frac{1}{2}$ total.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.

We have
$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$
where M is number of algorithm mistakes.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: continued.

$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$

m - best expert mistakes M algorithm mistakes.

$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$

Take log of both sides.

$-m \leq -M\log(4/3) + \log n.$

Solve for M.
$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$

Multiply by $1-\varepsilon$ for incorrect experts...

$(1-\varepsilon)^m \leq \left(1-\frac{\varepsilon}{2}\right)^M n.$

Massage...

$M \leq 2(1+\varepsilon)m + \frac{2\ln n}{\varepsilon}$

Approaches a factor of two of best expert performance!

Best Analysis?

Consider two experts: A,B

Bad example?

Which is worse?

(A) A correct even days, B correct odd days

(B) A correct first half of days, B correct second

Best expert peformance: $T/2$ mistakes.

Pattern (A): $T-1$ mistakes.

Factor of (almost) two worse!

Randomization!!!!

Better approach?

Use?

Randomization!

That is, choose expert i with prob $\propto w_i$

Bad example: A,B,A,B,A...

After a bit, A and B make nearly the same number of mistakes.

Choose each with approximately the same probabilty.

Make a mistake around $1/2$ of the time.

Best expert makes $T/2$ mistakes.

Roughly optimal!

Randomized analysis.

Some formulas:

For $\varepsilon \leq 1, x \in [0,1]$,

$(1+\varepsilon)^x \leq (1+\varepsilon x)$
$(1-\varepsilon)^x \leq (1-\varepsilon x)$

For $\varepsilon \in [0, \frac{1}{2}]$,

$-\varepsilon - \varepsilon^2 \leq \ln(1-\varepsilon) \leq -\varepsilon$
$\varepsilon - \varepsilon^2 \leq \ln(1+\varepsilon) \leq \varepsilon$

Proof Idea: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Randomized algorithm

Expert i loses $\ell_i^t \in [0,1]$ in round t.

1. Initially $w_i = 1$ for expert i.
2. Choose expert i with prob $\frac{w_i}{W}$, $W = \sum_i w_i$.
3. $w_i \leftarrow w_i(1-\varepsilon)^{\ell_i^t}$

$W(t)$ sum of w_i at time t. $W(0) = n$

Best expert, b, loses L^* total. $\rightarrow W(T) \geq w_b \geq (1-\varepsilon)^{L^*}$.

$L_t = \sum_i \frac{w_i \ell_i^t}{W}$ expected loss of alg. in time t.

Claim: $W(t+1) \leq W(t)(1-\varepsilon L_t)$ Loss $\rightarrow$ weight loss.

Proof:

$$\begin{aligned} W(t+1) = \sum_i (1-\varepsilon)^{\ell_i^t} w_i \leq \sum_i (1-\varepsilon \ell_i^t) w_i &= \sum_i w_i - \varepsilon \sum_i w_i \ell_i^t \\ &= \sum_i w_i \left(1 - \varepsilon \frac{\sum_i w_i \ell_i^t}{\sum_i w_i}\right) \\ &= W(t)(1-\varepsilon L_t) \end{aligned}$$

Analysis

$(1-\varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1-\varepsilon L_t)$

Take logs

$(L^*)\ln(1-\varepsilon) \leq \ln n + \sum \ln(1-\varepsilon L_t)$

Use $-\varepsilon - \varepsilon^2 \leq \ln(1-\varepsilon) \leq -\varepsilon$

$-(L^*)(\varepsilon + \varepsilon^2) \leq \ln n - \varepsilon \sum L_t$

And

$\sum_t L_t \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}$.

$\sum_t L_t$ is total expected loss of algorithm.

Within $(1+\varepsilon)$ ish of the best expert!

No factor of 2 loss!

Gains.

Why so negative?

Each day, each expert gives gain in $[0,1]$.

Multiplicative weights with $(1+\varepsilon)^{g_i^t}$.

$$G \geq (1-\varepsilon)G^* - \frac{\log n}{\varepsilon}$$

where G^* is payoff of best expert.

Scaling:

Not $[0,1]$, say $[0,\rho]$.

$$L \leq (1+\varepsilon)L^* + \frac{\rho \log n}{\varepsilon}$$

Summary: multiplicative weights.

Framework: n experts, each loses different amount every day.

Perfect Expert: $\log n$ mistakes.

Imperfect Expert: best makes m mistakes.

Deterministic Strategy: $2(1+\varepsilon)m + \frac{\log n}{\varepsilon}$

Real numbered losses: Best loses L^* total.

Randomized Strategy: $(1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

Strategy:

Choose proportional to weights
multiply weight by $(1-\varepsilon)^{\text{loss}}$.

Multiplicative weights framework!

Applications next!

Two person zero sum games.

$m \times n$ payoff matrix A.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair (x,y):

$$p(x,y) = x^t A y$$

That is,

$$\sum_i x_i \left(\sum_j a_{i,j} y_j\right) = \sum_j \left(\sum_i x_i a_{i,j}\right) y_j.$$

Recall row minimizes, column maximizes.

Equilibrium pair: (x^*, y^*)?

$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

Equilibrium.

Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t Ay^*$. [1]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t Ay^*$.

[1] $A^{(i)}$ is ith row.

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t Ay).$$

Note: x can be $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Agin: y of form $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of C?

Duality.

$$R = \max_y \min_x (x^t Ay).$$
$$C = \min_x \max_y (x^t Ay).$$

Weak Duality: $R \leq C$.
Proof: Better to go second. □

At Equilibrium (x^*, y^*), payoff v:
row payoffs (Ay^*) all $\geq v \implies R \geq v$.
column payoffs $((x^*)^t A)$ all $\leq v \implies v \geq C$.
$\implies R \geq C$

Equilibrium $\implies R = C$!

Strong Duality: There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

Proof of Equilibrium.

Later. Well in just a minute.....

Aproximate equilibrium ...

$C(x) = \max_y x^t Ay$

$R(y) = \min_x x^t Ay$

Always: $R(y) \leq C(x)$

Strategy pair: (x,y)

Equilibrium: (x,y)

$R(y) = C(x) \rightarrow C(x) - R(y) = 0$.

Approximate Equilibrium: $C(x) - R(y) \leq \varepsilon$.

With $R(y) < C(x)$
→ "Response y to x is within ε of best response"
→ "Response x to y is within ε of best response"

Proof of approximate equilibrium.

How?

(A) Using geometry.

(B) Using a fixed point theorem.

(C) Using multiplicative weights.

(D) By the skin of my teeth.

(C) ..and (D).
Not hard. Even easy. Still, head scratching happens.

Games and experts

Again: find (x^*, y^*), such that

$(\max_y x^* Ay) - (\min_x x^* Ay^*) \leq \varepsilon$

$C(x^*) \quad - \quad R(y^*) \leq \varepsilon$

Experts Framework:
n Experts, T days, L^* -total loss of best expert.

Multiplicative Weights Method yields loss L where

$L \leq (1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

Games and Experts.

Assume: A has payoffs in $[0,1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) m pure row strategies are experts.
Use multiplicative weights, produce row distribution.
Let x_t be distribution (row strategy) on day t.

2) Each day, adversary plays best column response to x_t.
Choose column of A that maximizes row's expected loss.
Let y_t be indicator vector for this column.

Comments

For any ε, there exists an ε-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here?
Limit of a sequence on some closed set..hmmm..

Later: will use geometry, linear programming.

Complexity?
$T = \frac{\ln n}{\varepsilon^2} \to O(nm\frac{\log n}{\varepsilon^2})$. Basically linear!

Versus Linear Programming: $O(n^3 m)$ Basically quadratic.
(Faster linear programming: $O(\sqrt{n+m})$ linear system solves.)
Still much slower ... and more complicated.

Dynamics: best response, update weight, best response.

Also works with both using multiplicative weights.

"In practice."

Approximate Equilibrium!

Experts: x_t is strategy on day t, y_t is best column against x_t.

Let $y^* = \frac{1}{T}\sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

Claim: (x^*, y^*) are 2ε-optimal for matrix A.

Column payoff: $C(x^*) = \max_y x^* Ay$.
Loss on day t, $x_t A y_t \geq C(x^*)$ by the choice of x^*.
Thus, algorithm loss, L, is $\geq T \times C(x^*)$.

Best expert: L^*- best row against all the columns played.

best row against $\sum_t Ay_t$ and $T \times y^* = \sum_t y_t$
$\to$ best row against $T \times Ay^*$.
$\to L^* \leq T \times R(y^*)$.

Multiplicative Weights: $L \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$T \times C(x^*) \leq (1+\varepsilon)T \times R(y^*) + \frac{\ln n}{\varepsilon} \to C(x^*) \leq (1+\varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T}$
$\to C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$.

$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1$
$\to C(x^*) - R(y^*) \leq 2\varepsilon$.

Homework 2 out this week.

See you on Thursday.

Approximate Equilibrium: slightly different!

Experts: x_t is strategy on day t, y_t is best column against x_t.

Let $x^* = \frac{1}{T}\sum_t x_t$ and $y^* = \frac{1}{T}\sum_t y_t$.

Claim: (x^*, y^*) are 2ε-optimal for matrix A.

Column payoff: $C(x^*) = \max_y x^* Ay$.
Let y_r be best response to $C(x^*)$.
Day t, $x_t A y_t \geq x_t A y_r$ – y_t is best response to x_t.
Algorithm loss: $\sum_t x_t A y_t \geq \sum_t x_t A y_r$
$L \geq T \times C(x^*)$.

Best expert: L^*- best row against all the columns played.

best row against $\sum_t Ay_t$ and $Ty^* = \sum_t y_t$
$\to$ best row against TAy^*.
$\to L^* \leq T \times R(y^*)$.

Multiplicative Weights: $L \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$TC(x^*) \leq (1+\varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \to C(x^*) \leq (1+\varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T}$
$\to C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$.

$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1 \to C(x^*) - R(y^*) \leq 2\varepsilon$.