An information-theoretic view of the Witsenhausen Counterexample

Anant Sahai
presenting joint work with students:
Pulkit Grover   Se Yong Park

Wireless Foundations Center
U.C. Berkeley

Support from the National Science Foundation and Samsung

2009 ITA in UCSD, La Jolla

Ho, Kastner, Wong '78

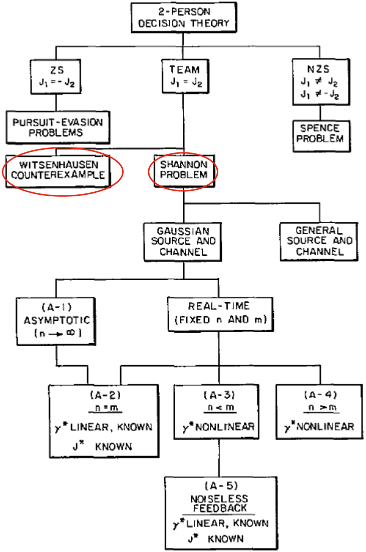
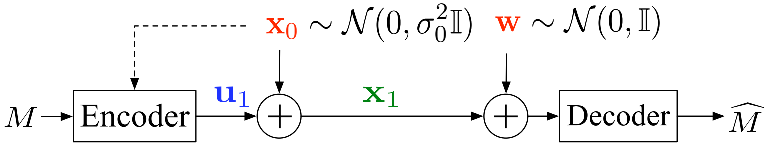


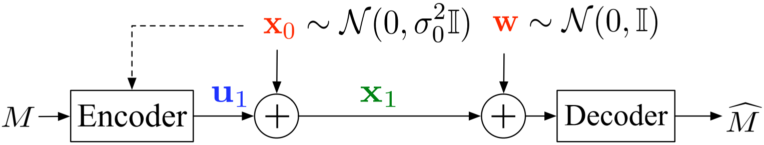
Fig. 1. Teams, signaling, and information theory.

Costa '83



Dirty paper coding

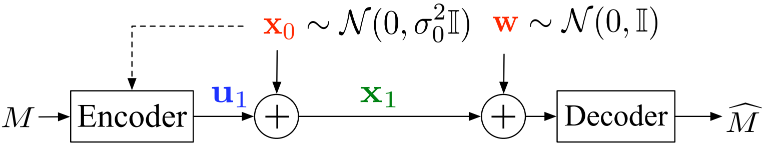
Kim, Sutivong, Cover '08



Dirty paper coding

- State Amplification: also estimate $\mathbf{x}_0$ (inspired by Devroye, *et al '06* and following up on Sutivong, *et al* '05)

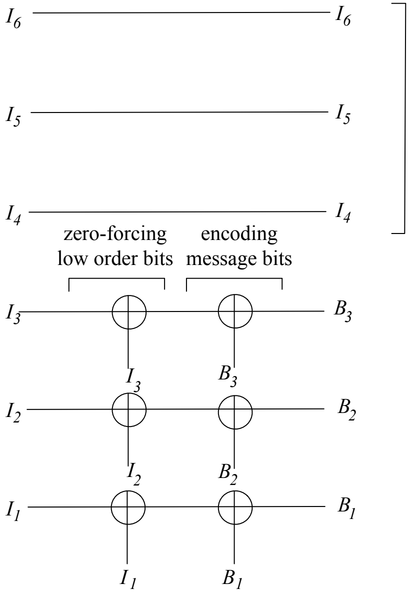
Merhav and Shamai '07



Dirty paper coding

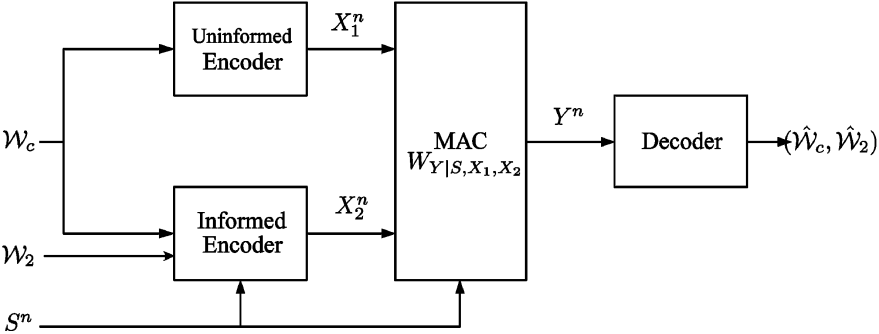
- State Amplification: also estimate $\mathbf{x}_0$ (inspired by Devroye, *et al '06* and following up on Sutivong, *et al* '05)
- State Masking: obscure $\mathbf{x}_0$

Deterministic perspective: can DPC be distributed?

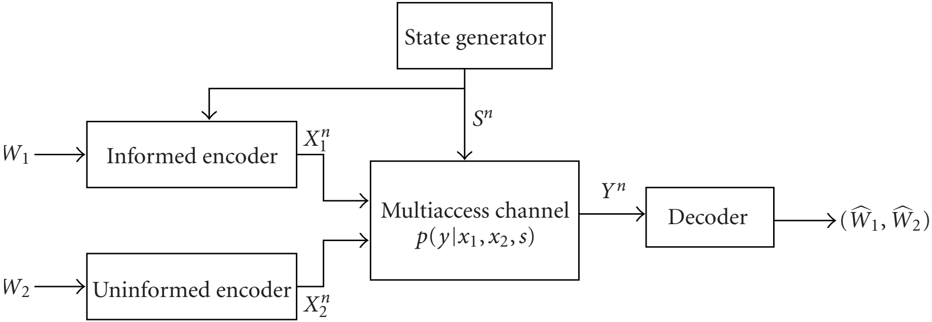


The most significant bits of the interference are untouched

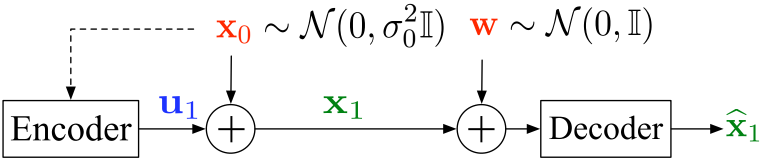
Somekh-Baruch, Shamai, Verdú '08



Kotagiri and Laneman '08



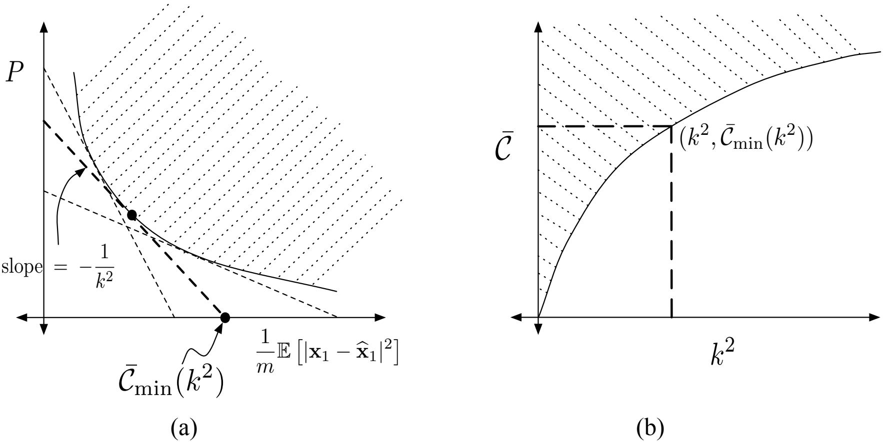
Simplify: eliminate both messages!



$$\mathbb{E}[\mathcal{C}] = k^2 P + \frac{1}{m} \mathbb{E}\left[\|\mathbf{x}_1 - \widehat{\mathbf{x}}_1\|^2\right]$$

The vector Witsenhausen counterexample

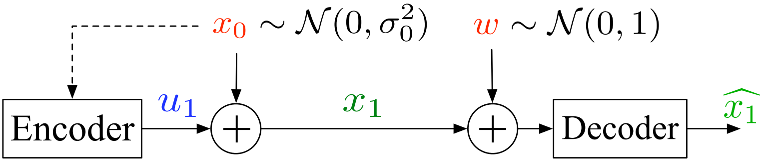
Two equivalent perspectives



Outline

- Towards the "simplest" unsolved problem
- **The perspective from control theory**
- A new bound and an approximately optimal solution
- Approximate optimality in the scalar case

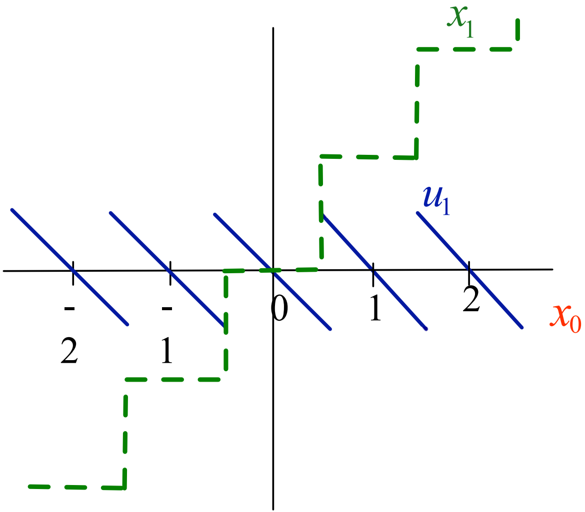
Witsenhausen '68



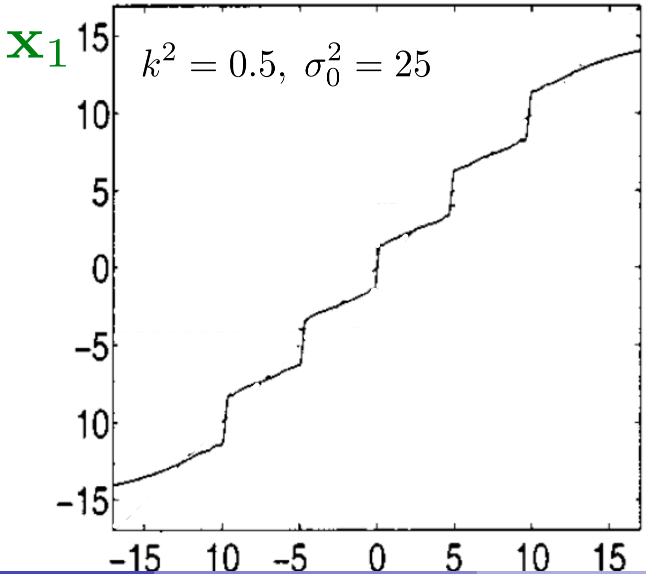
The scalar Witsenhausen counterexample

$$\mathcal{C} = k^2 {u_1}^2 + (x_1 - \widehat{x}_1)^2$$

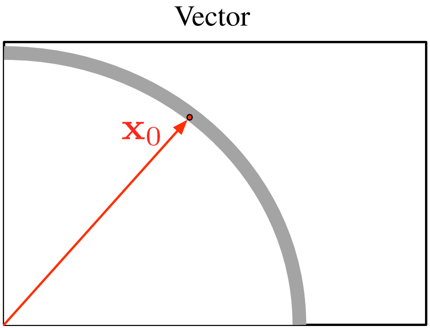
Mitter and Sahai '99

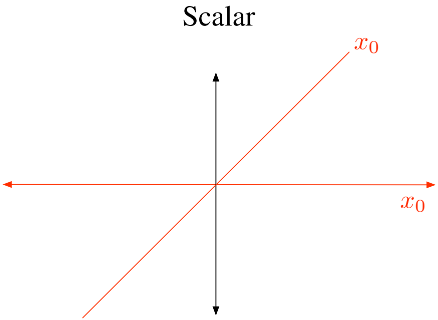


Baglietto, Parisini, Zoppoli '97

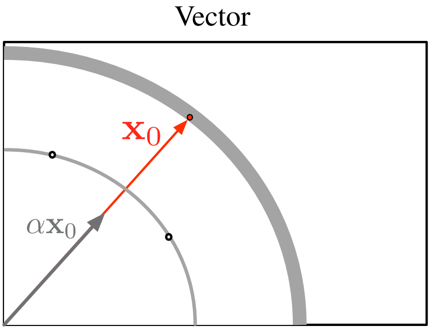


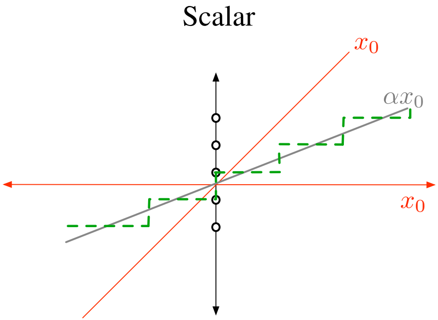
The connection with DPC





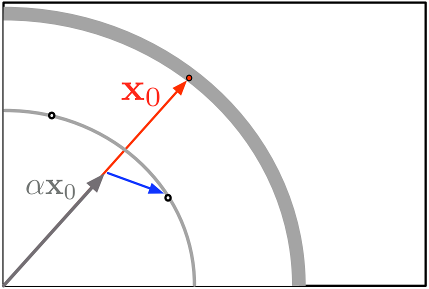
The connection with DPC



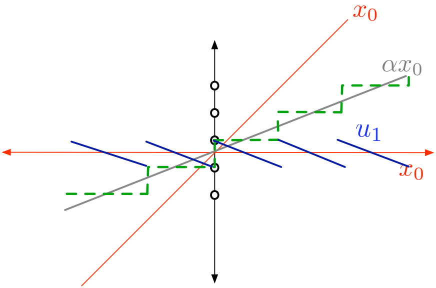


The connection with DPC

Vector

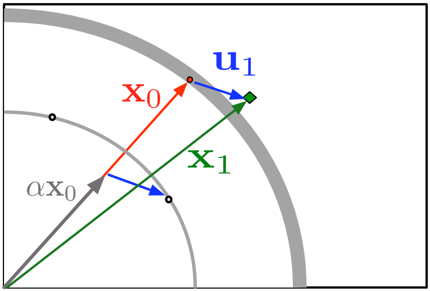


Scalar

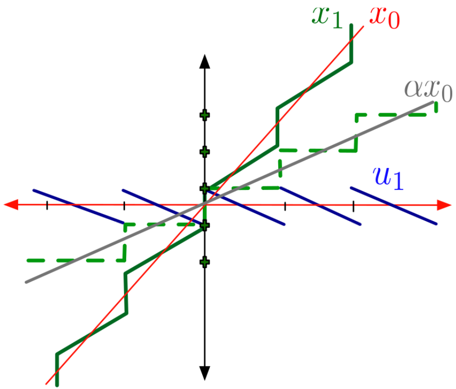


The connection with DPC

Vector

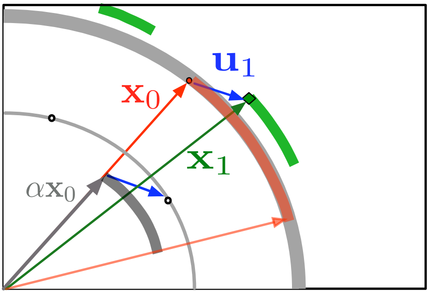


Scalar

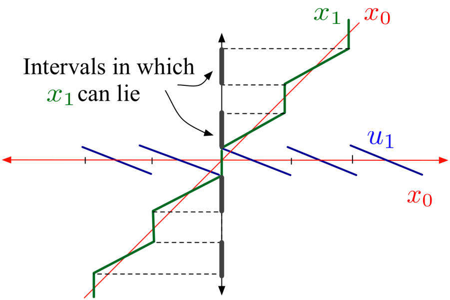


The connection with DPC

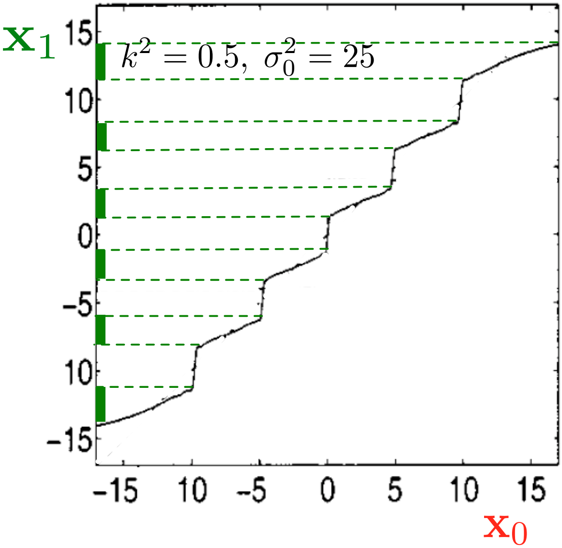
Vector



Scalar



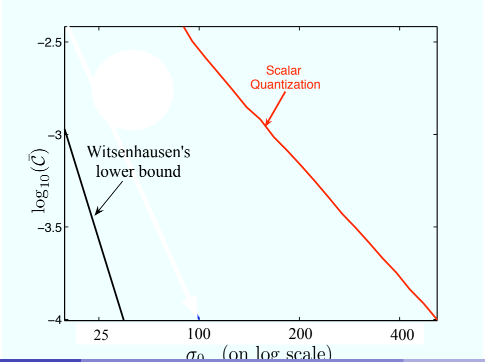
Baglietto, Parisini, Zoppoli '97



Witsenhausen's scalar lower bound

$$\bar{\mathcal{C}}_{\min}^{\text{scalar}} \geq \frac{1}{\sigma_0} \int_{-\infty}^{+\infty} \phi\left(\frac{\xi}{\sigma_0}\right) V_k(\xi) d\xi,$$

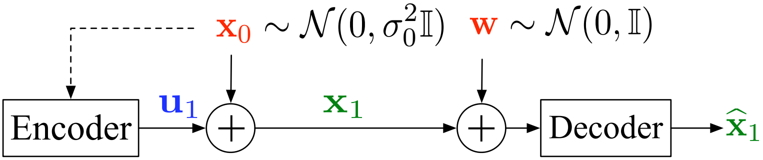
where $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})$, $V_k(\xi) := \min_a [k^2(a-\xi)^2 + h(a)]$, and
$h(a) := \sqrt{2\pi} a^2 \phi(a) \int_{-\infty}^{+\infty} \frac{\phi(y)}{\cosh(ay)} dy.$



Outline

- Towards the “simplest” unsolved problem
- The perspective from control theory
- **A new bound and an approximately optimal solution**
- Approximate optimality in the scalar case

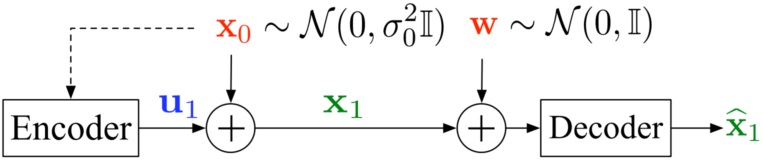
The core tension



$$\mathbb{E}[\mathcal{C}] = k^2 P + \frac{1}{m}\mathbb{E}\left[\|\mathbf{x}_1 - \widehat{\mathbf{x}}_1\|^2\right]$$

The vector Witsenhausen counterexample

The new bound



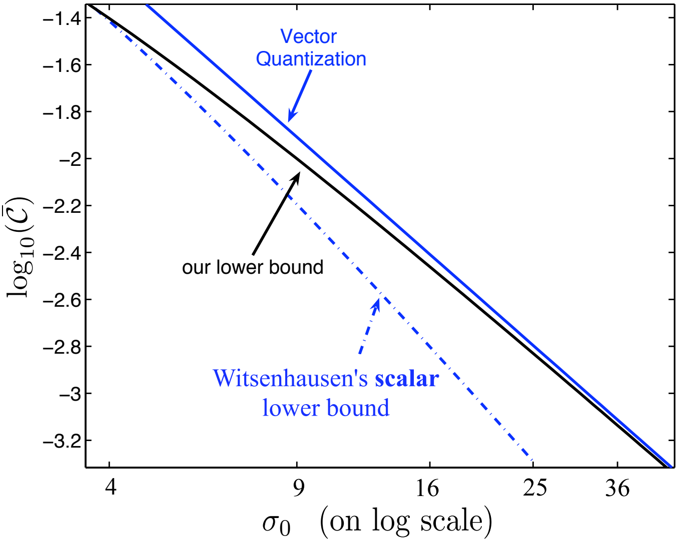
$$\mathbb{E}[\mathcal{C}] = k^2 P + \frac{1}{m} \mathbb{E}\left[\|\mathbf{x}_1 - \widehat{\mathbf{x}}_1\|^2\right]$$

The vector Witsenhausen counterexample

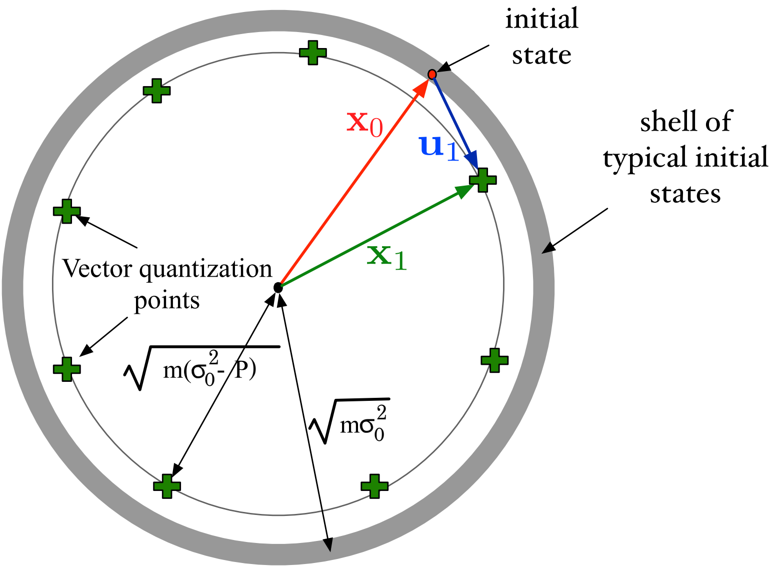
$$\bar{\mathcal{C}}_{\min} \geq \inf_{P \geq 0} k^2 P + \left(\left(\sqrt{\kappa(P)} - \sqrt{P}\right)^+\right)^2,$$

$$\text{where } \kappa(P) = \frac{\sigma_0^2}{\sigma_0^2 + 2\sigma_0\sqrt{P} + P + 1}.$$

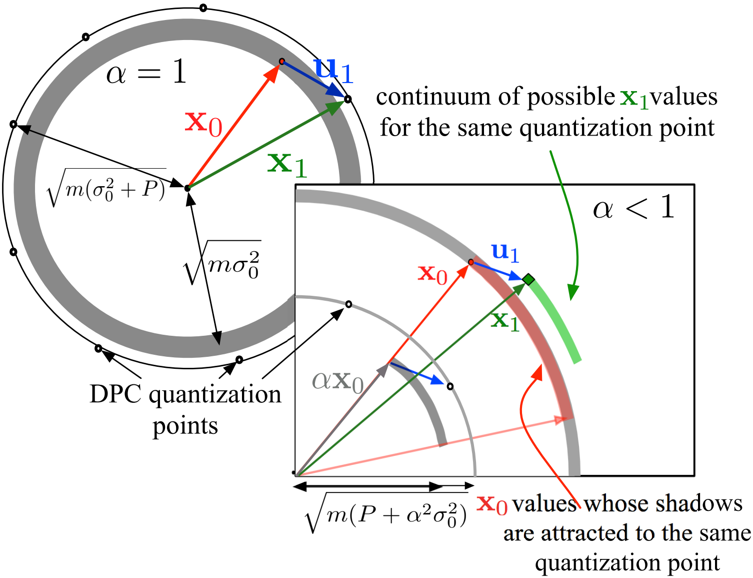
Comparison with old bound



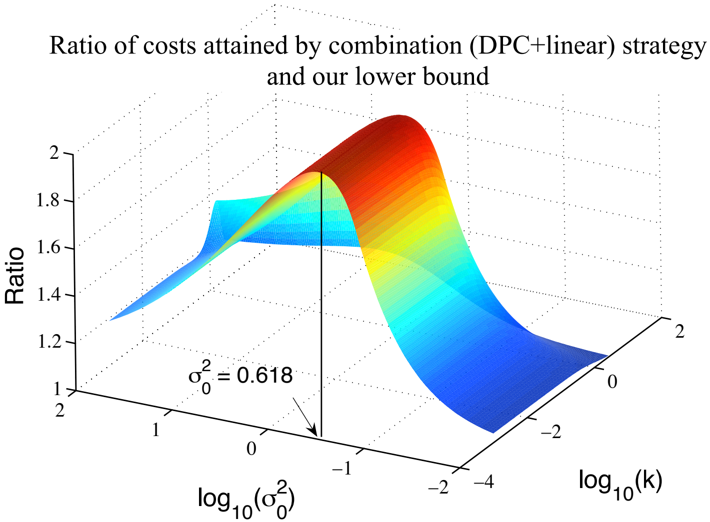
Achievability



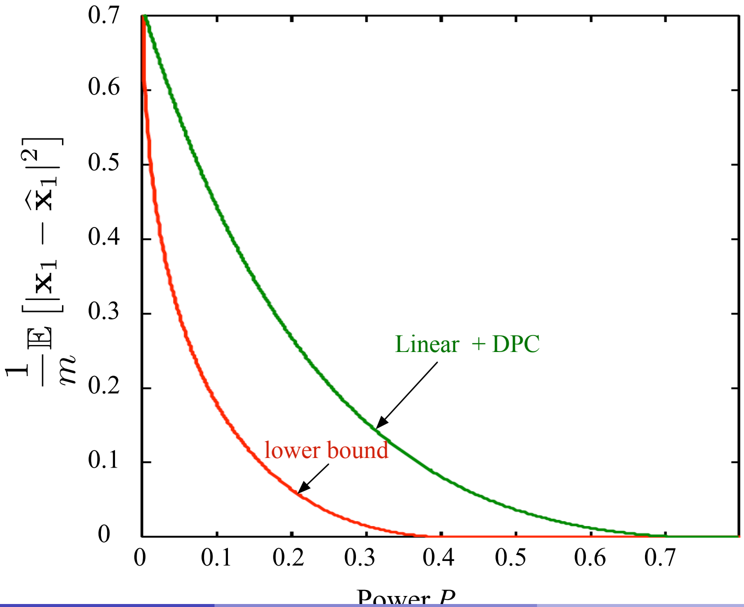
Achievability



Constant factor bound



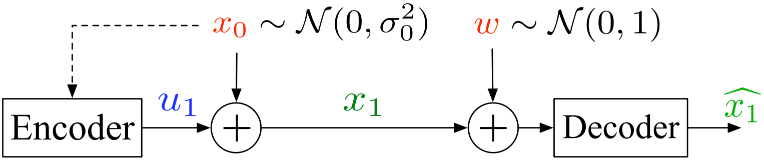
Constant factor bound



Outline

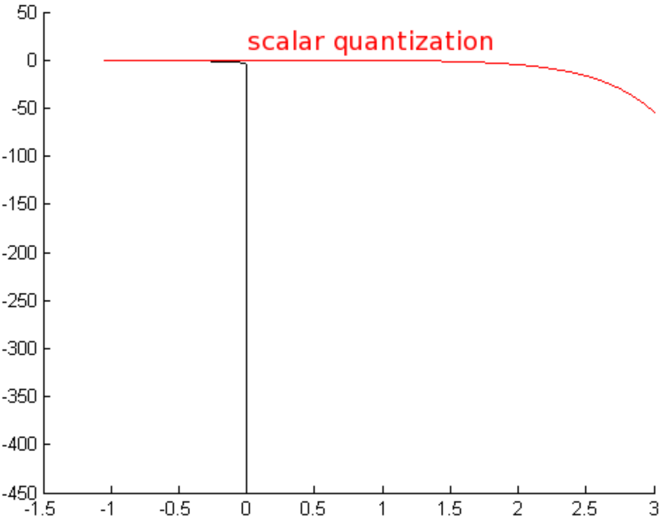
- Towards the "simplest" unsolved problem
- The perspective from control theory
- A new bound and an approximately optimal solution
- **Approximate optimality in the scalar case**

Back to the control setting

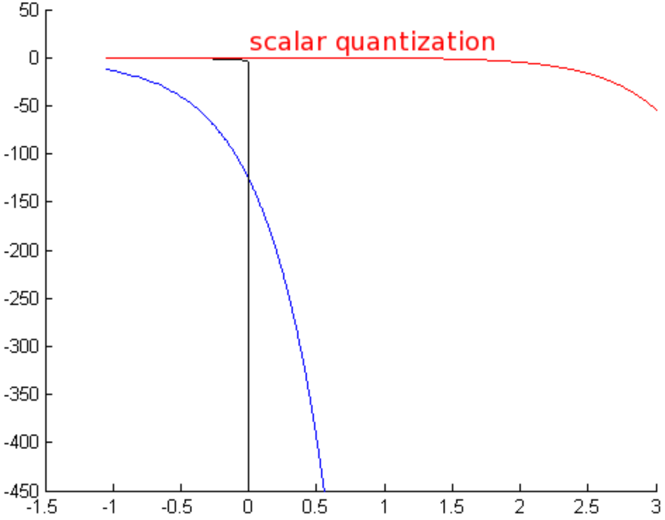


The scalar Witsenhausen counterexample

Back to the control setting



Back to the control setting



Conclusions

- The Witsenhausen counterexample is the “simplest” unsolved problem in information theory.
- Standard tools give a constant-factor result in asymptopia
- We need non-asymptotic approximation guarantees too