Trade-off of lossless source coding error exponents

Cheng Chang   Anant Sahai

HP Labs, Palo Alto
EECS, UC Berkeley

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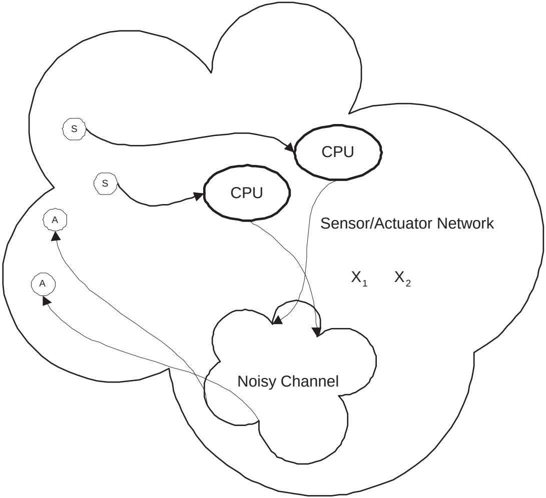
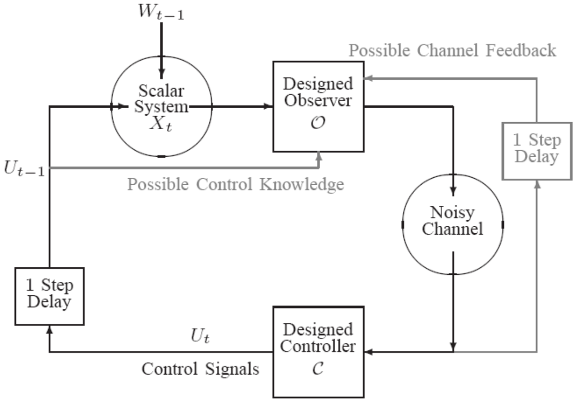
Motivation

- Sensor networks: control, communication and information theory
  - Distributed control
  - Unstable process with noisy feedback loop



Stabilizing an unstable system with noisy feedback

- Scalar case $X_{t+1} = \lambda X_t + U_t + W_t$ (Sahai/Mitter 2006)
  - $E(|X_t|^\eta) < \infty$, for all t iff delay exponent $E(\log_2(\lambda)) > \eta \log_2(\lambda)$



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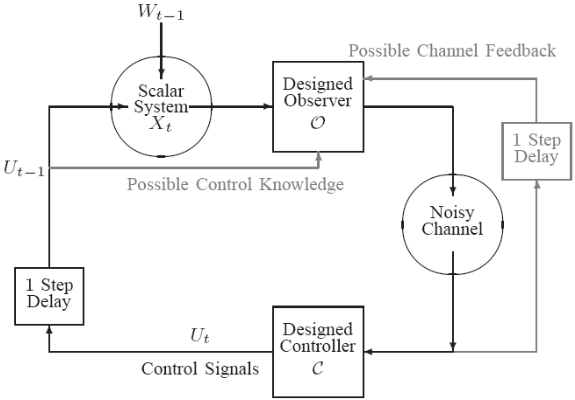


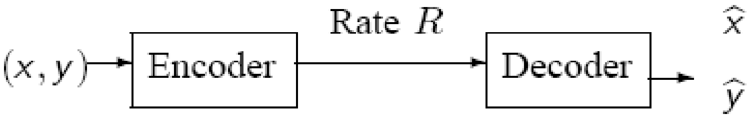
- Vector case $\vec{X}_{t+1} = A\vec{X}_t + \vec{U}_t + \vec{W}_t$: A eigenvalues $\lambda_1, \lambda_2 \ldots$
- Different error exponents for different streams

Related work

- Messages requiring different reliability

- Sub-messages requiring different reliability

- This talk is about multi-stream trade-offs.



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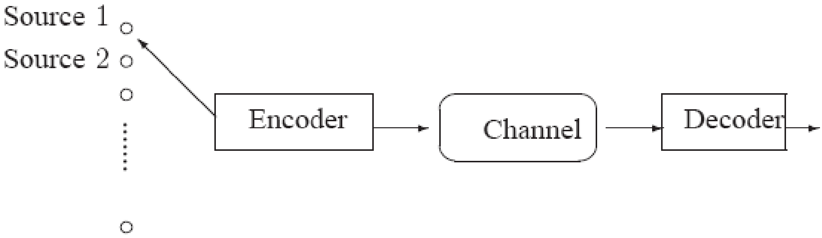

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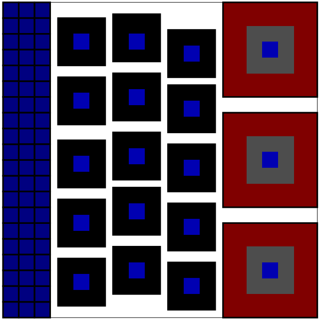

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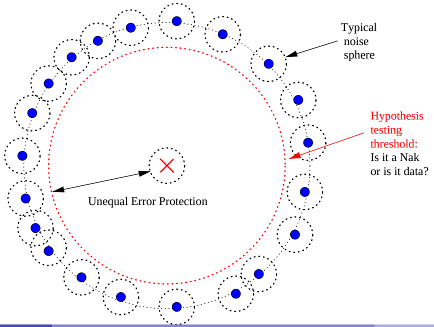

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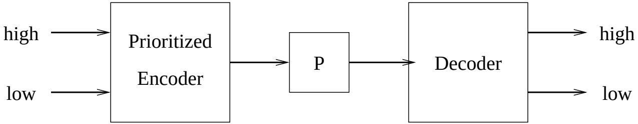

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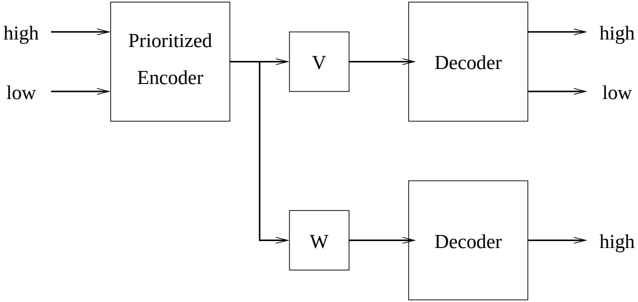

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- Sub-messages requiring different reliability
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  - Multiterminal coding error-exponent trade-offs (Weng *et al* ’08, Kaspi *et al* ’08, Etkin *et al* ’08)
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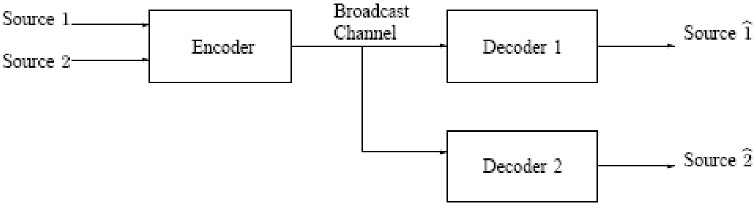

Outline

1. Motivation and related work
2. Main Results:
   - Trade-off for block source coding: a complete characterization
   - Trade-off for streaming source coding: inner and outer bounds
   - Trade-off for BEC with feedback and its control implications
3. Conclusions

Block source-coding error exponents

- Fixed-length block coding for i.i.d. source $x \sim P$

$$x_1^n \in \mathcal{X}^n \rightarrow b_1^{nR} \in \{0,1\}^{nR} \rightarrow \widehat{x}_1^n \in \mathcal{X}^n$$

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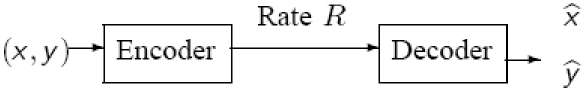
- Entropy and error exponent :
  - $R > H(P)$
  - $\Pr(\widehat{x}_1^n \neq x_1^n) \sim 2^{-nE_{block}(R)}$ for optimal coding

$$E_{block}(R) = \min_{Q:H(Q)\geq R} D(Q\|P)$$

Error-exponent region (block coding)

- Two sources X and Y share R bits/sec



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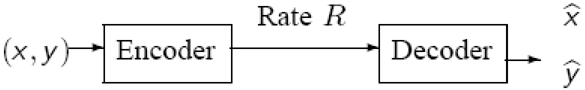

- Error-exponent region
$\{(E_x, E_y) : \Pr(\hat{x}_1^n \neq x_1^n) \sim 2^{-nE_x}$ and $\Pr(\hat{y}_1^n \neq y_1^n) \sim 2^{-nE_y}\}$ for some coding scheme

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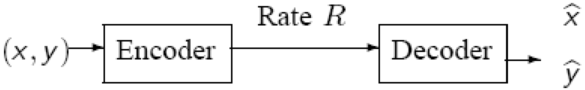

- Error-exponent region $\{(E_x, E_y) : \Pr(\hat{x}_1^n \neq x_1^n) \sim 2^{-nE_x} \text{ and } \Pr(\hat{y}_1^n \neq y_1^n) \sim 2^{-nE_y}\}$ for some coding scheme
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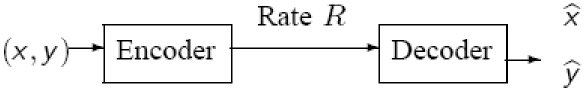

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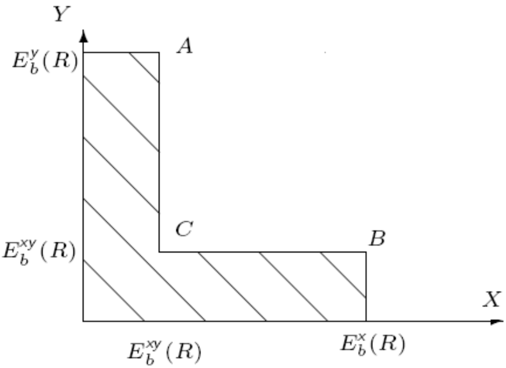
- Naive outer bound (converse):
  - $E_x \leq E_{block(x)}(R)$, all for one source
  - Either $E_x \leq E_{block(xy)}(R)$ or $E_y \leq E_{block(xy)}(R)$

Block bound illustrated



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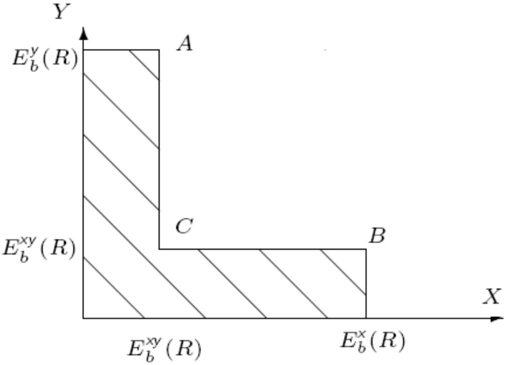

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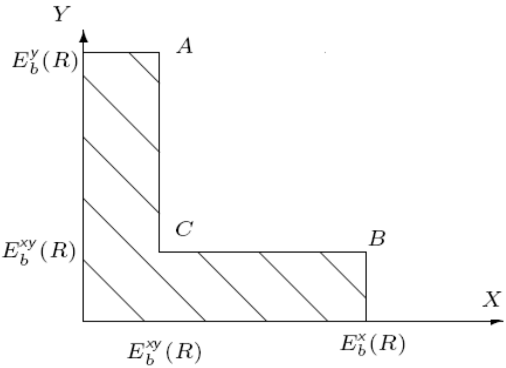

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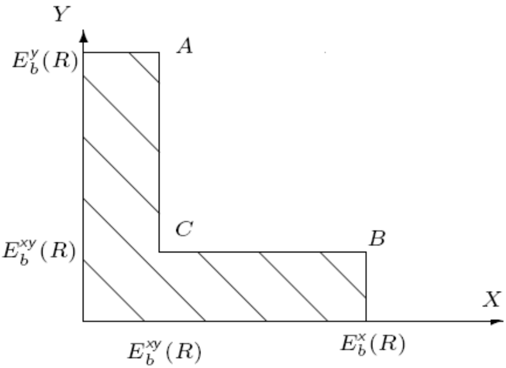

- A non-convex region.
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  - Transmit high priority source if cannot send both

Beyond block coding

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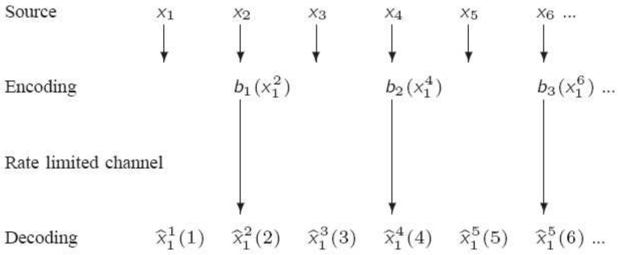

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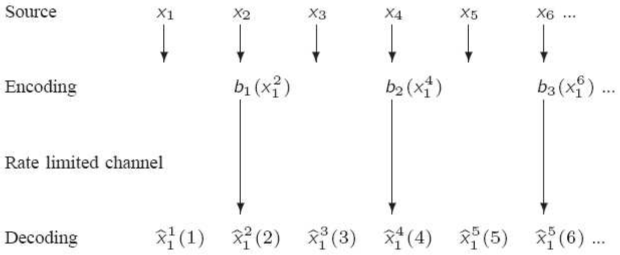

  - Symbol-wise decoding error: $\Pr(x_i \neq \widehat{x}_i(i+\Delta)) \sim 2^{-\Delta E(R)}$
    Focusing bound $E(R) = \inf_{\alpha>0} \frac{1}{\alpha} E_{block}((\alpha+1)R)$

Error-exponent region (streaming)

- Two streaming sources X and Y share R bits/sec
- Error exponent region

$$\{(E_x, E_y) : \Pr(x_t \neq \widehat{x}_t(t+\Delta)) \sim 2^{-\Delta E_x} \text{ and } \Pr(y_t \neq \widehat{y}_t(t+\Delta)) \sim 2^{-\Delta E_y}\}$$

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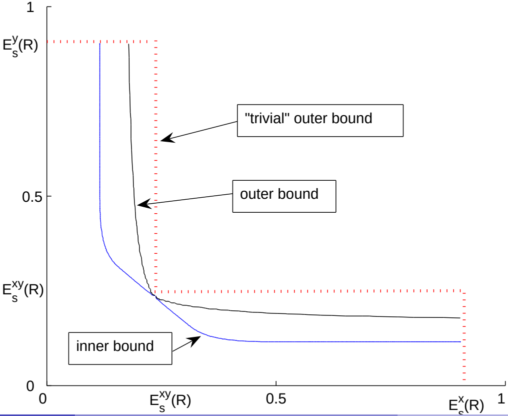
- Inner and outer bounds



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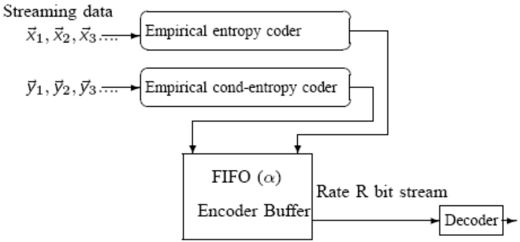

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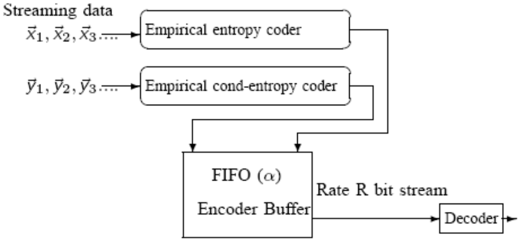
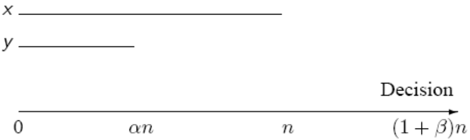


- Outer bound: uncertainty-focusing bound for two sources



Binary-erasure channel with feedback exponent region

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Binary-erasure channel with feedback exponent region

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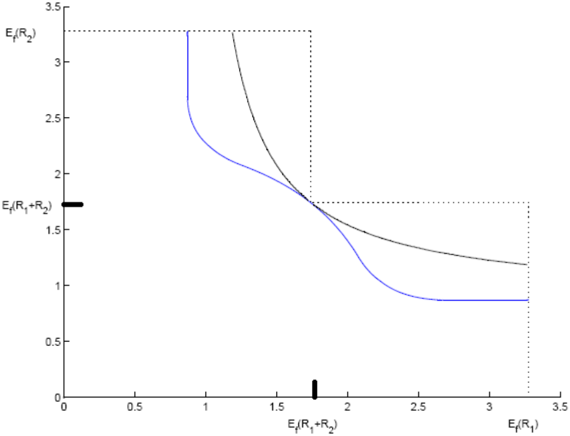
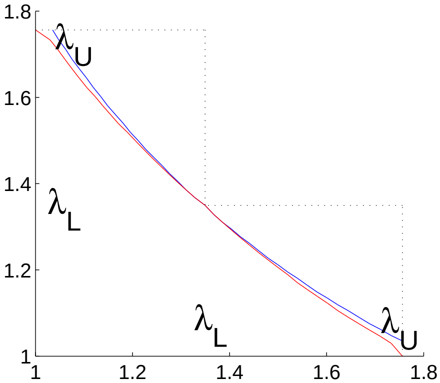
- Generalization of point to point case (Sahai '08)
- Inner bounds (better than Sahai '00) and outer bounds



Implications for control systems

- $\vec{X}_{t+1} = A\vec{X}_t + \vec{U}_t + \vec{W}_t$: A with eigenvalues λ_1, λ_2
- Control-feedback channel: BEC with feedback
- Inner and outer bounds on stabilizable(λ_1, λ_2) for $\beta = 0.1, \eta = 2$



- Related to Minero, *et al* ('08).

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  - Sufficient/Necessary (not both) conditions for stabilizing vector processes through a BEC feedback channel
- Future directions
  - Tightening the inner and outer bounds
  - Extending to general channels
  - Trade-off of lossy source-coding error exponents (block and streaming)