Universal Anytime Codes: An approach to uncertain channels in control

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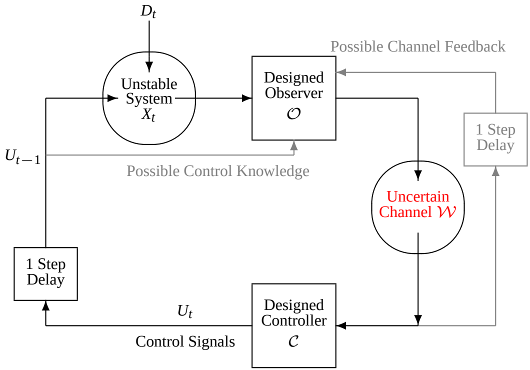
Outline

1. Problem setup
   - Channel uncertainty and stabilization
   - Review of past results
2. New result: universal anytime codes
3. Sufficient condition for stabilization
4. Conclusion

Our simple distributed control problem



$$X_{t+1} = \lambda X_t + U_t + D_t$$

- Unstable $\lambda > 1$, bounded initial condition and disturbance D.

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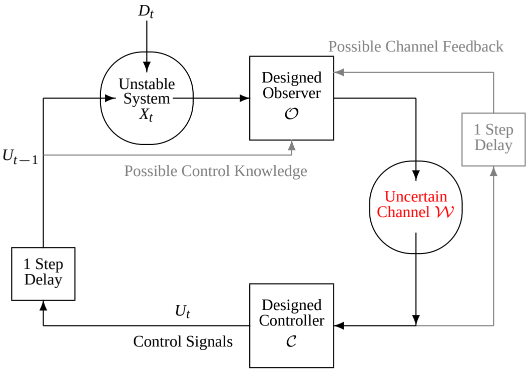

$$X_{t+1} = \lambda X_t + U_t + D_t$$

- Unstable $\lambda > 1$, bounded initial condition and disturbance D.
- Goal: η-stabilization $\sup_{t>0} E[\|X_t\|^\eta] \le K$ for some $K < \infty$.

Model of Channel Uncertainty: “Compound Channels”

- Channel W is known to be memoryless across time.
- Input and output alphabets are known $(\mathcal{Y}, \mathcal{Z})$ and finite.
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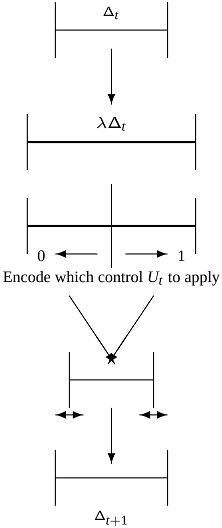
  - Nature can choose any particular $W \in \mathcal{W}$
  - Choice remains fixed for all time.
- Capacity well understood: $C = \sup_Q \inf_{W \in \mathcal{W}} I(Q, W)$

Review: Entirely noiseless channel



Window known to contain X_t

will grow by factor of $\lambda > 1$

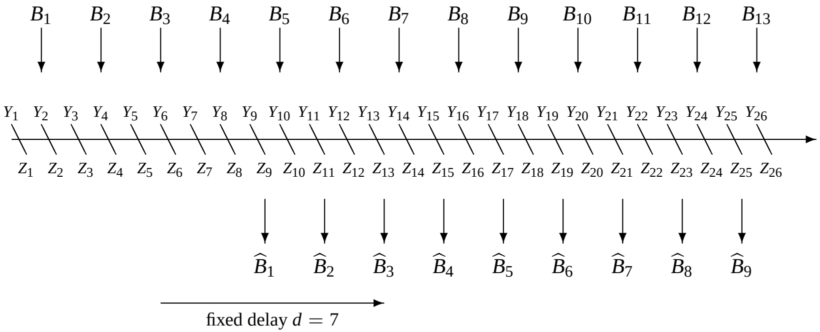
Sending R bits, cut window by a factor of 2^{-R}

grows by $\frac{\Omega}{2}$ on each side

giving a new window for X_{t+1}

As long as $R > \log_2 \lambda$, we can have Δ stay bounded forever.

Review: Delay-universal (*anytime*) communication



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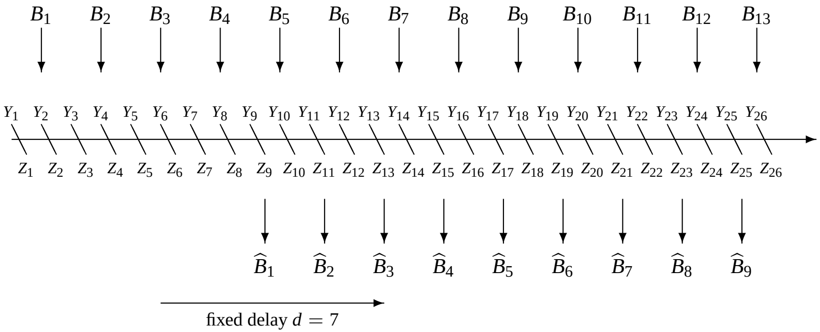

- Fixed-delay reliability α is achievable if there exists a sequence of encoder/decoder pairs with increasing end-to-end delays d_j such that $\lim_{j\to\infty} \frac{-1}{d_j} \ln P(B_i \neq \hat{B}_i^j) = \alpha$.

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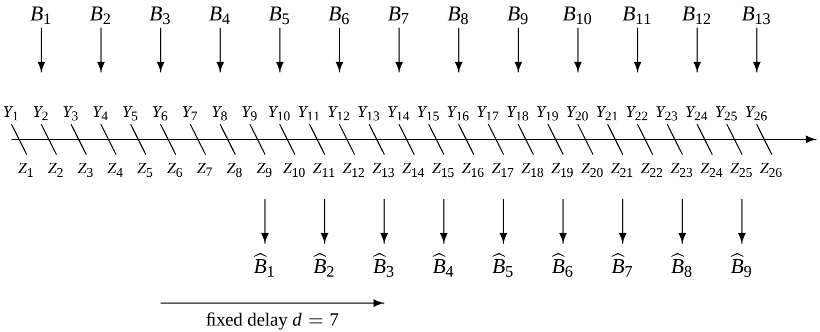

- Reliability α is achievable *delay-universally* or in an *anytime fashion* if a single encoder works for all sufficiently large delays d.

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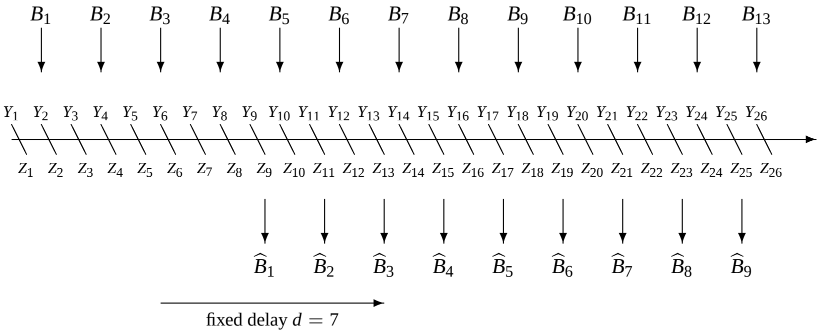

- The anytime capacity $C_{\text{any}}(\alpha)$ is the supremal rate at which reliability α is achievable in a delay-universal way.

Review: Separation theorem for scalar control

Necessity: If a scalar system with parameter $\lambda > 1$ can be stabilized with finite η-moment across a noisy channel, then the **channel with noiseless feedback** must have

$$C_{\text{any}}(\eta \ln \lambda) \geq \ln \lambda$$

In general: If $P(|X| > m) < f(m)$, then $\exists K : P_{\text{error}}(d) < f(K\lambda^d)$

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Sufficiency: If there is an $\alpha > \eta \ln \lambda$ for which the **channel with noiseless feedback** has

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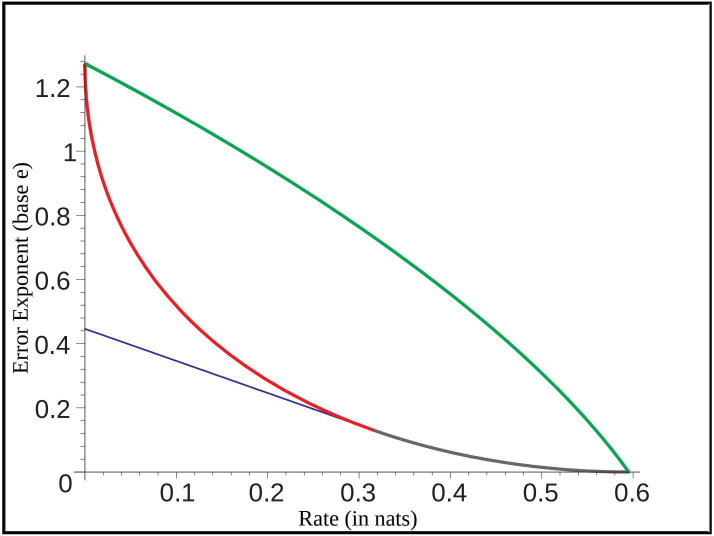
Proved using a direct equivalence so it also holds for compound channels.

From Rate/Reliability to Gain/Moments



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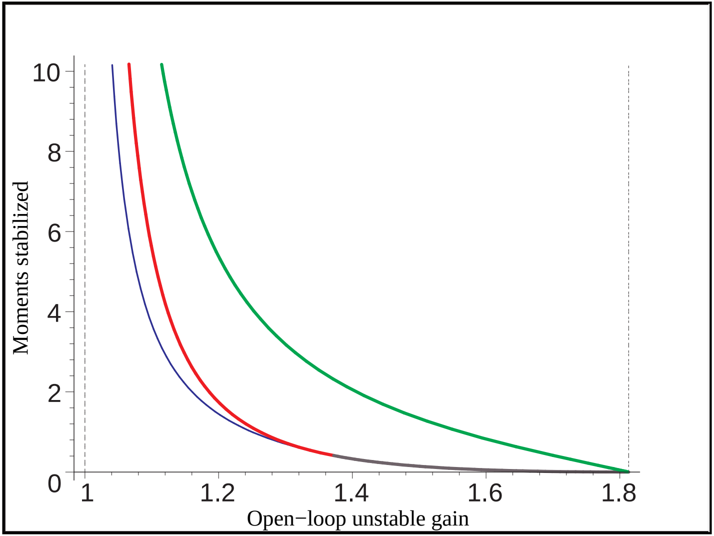
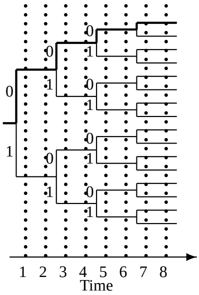


Why the random-coding bound works: tree codes



Tree with iid random labels:

- Data chooses a path through the tree
- Transmit the path labels through the channel
- Feedback is not used

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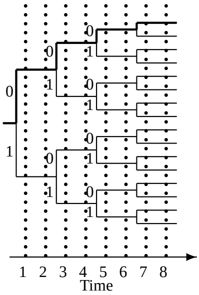

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  - Disjoint segments are pairwise independent of the true path.
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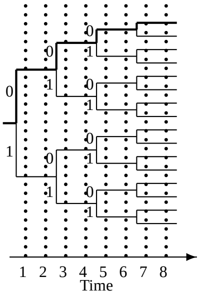

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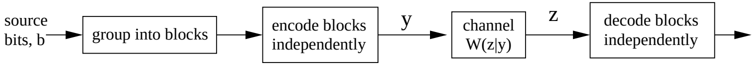
  - **Log likelihoods add along path.**
  - Disjoint segments are pairwise independent of the true path.
  - $E_r(R)$ analysis applies at the suffix
  - Shortest suffix dominates
- **Achieves** $P_e(d) \leq K \exp(-E_r(R)d)$ **for every** d **for all** $R < C$

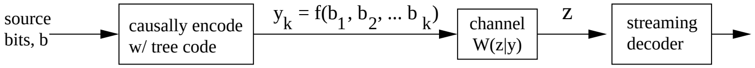
Relevant aspects of block and anytime codes

Block coding:



- Rare blocks in error
- Erroneous blocks never recovered
- Eventually results in exponential instability

Anytime coding:



- Can revisit earlier bit estimates
- Estimate reliabilities increase with delay
- Eventually detect and compensate for any incorrect controls

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  - The random-coding error exponent $E_r(R, Q, W)$ is achieved.

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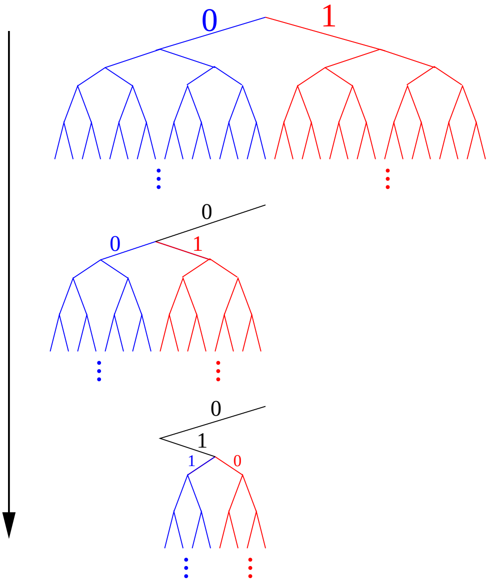
  - Ex: (0101,0101) + (1010,0101) gives $4 + 4 \neq 0$
  - The polynomial term grows with n not with delay d
  - Delays must be longer as time goes on.

Sequential Suffix-EMI Decoding

Decode sequentially, using max empirical mutual inform. (EMI) comparisons at each stage



If a blue codeword has the max EMI, decode first bit to 0, else to 1

If a blue codeword suffix has the max EMI, decode second bit to 0, else to 1

If a blue codeword suffix has the max EMI, decode third bit to 0, else to 1

The universal anytime coding theorem

Given a rate $R > 0$ and a compound discrete memoryless channel $\mathcal{W}$, there exists a random anytime code such that for all $E < E_{\text{any,univ}}(R)$ there is a constant $K > 0$ such that $\Pr[\widehat{B}_{n-d} \neq B_{n-d}] \leq K2^{-dE}$ for all $n, d \geq 0$ where

$$\begin{aligned} E_{\text{any,univ}}(R) &= \sup_{Q} \inf_{W \in \mathcal{W}} \inf_{P,V} D(P \times V \| Q \times W) + \max\{0, I(P,V) - R\} \\ &= \sup_{Q} \inf_{W \in \mathcal{W}} E_r(R, Q, W) \end{aligned}$$

- No feedback is needed
- Does as well as could be hoped for — essentially hits $E_r(R)$

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Back to the stabilization problem

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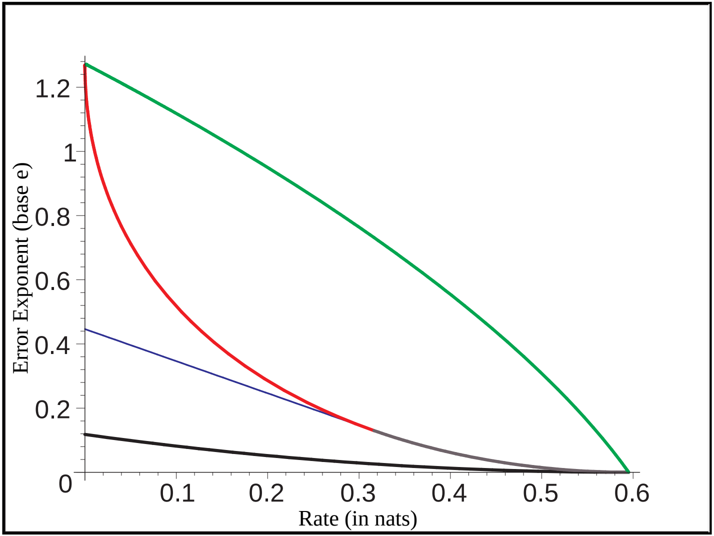
  - Translate to η-stabilization:

$$C(\mathcal{W}) - \log \lambda > (2 \log |\mathcal{Z}|) \sqrt{\frac{\eta \log \lambda}{\log e} (1 + \frac{2(\log e)^2}{e^2 (\log |\mathcal{Z}|)^2})}$$

  - Says that $O(\sqrt{\eta \log \lambda})$ extra capacity suffices to get η-th moment stabilized.

Visualizing the new universal sufficient condition



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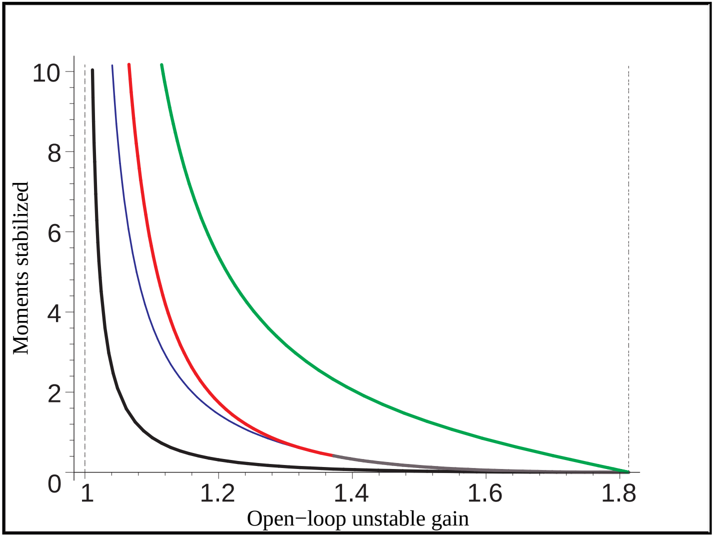

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