New results on source and channel coding error exponents with respect to end-to-end delay

The power of "punctuation"

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Wireless Foundations
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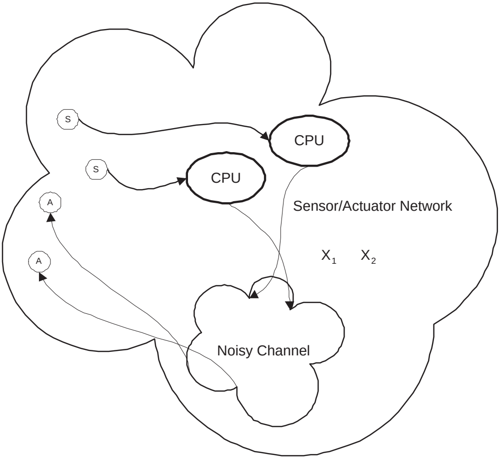
Asilomar 2006

Motivation

- Many applications are embedded in time



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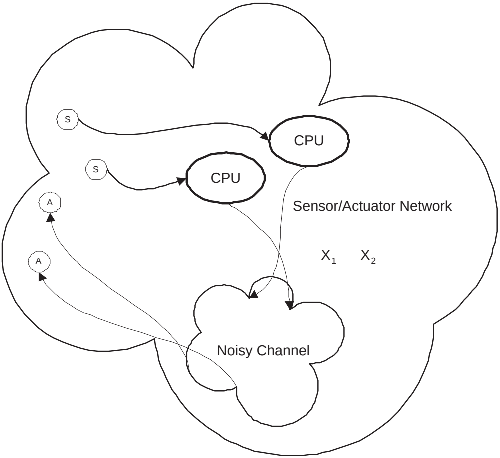

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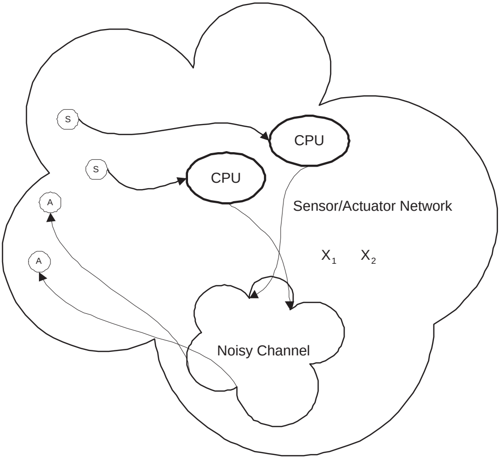

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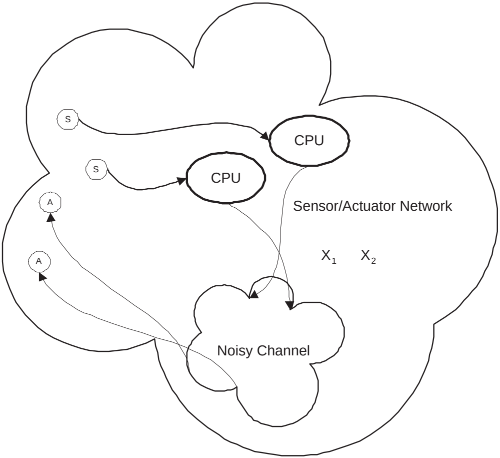
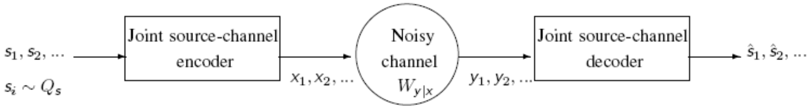


Simplest problem: point-to-point

- Lossless source-channel coding (synchronized)
  - One source symbol per channel use



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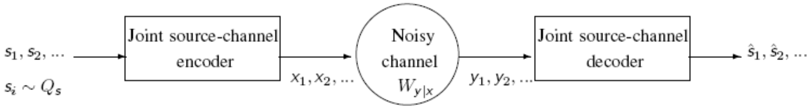

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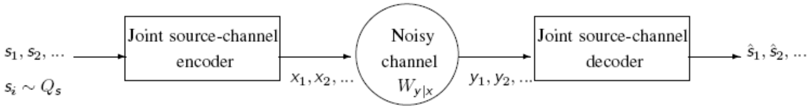

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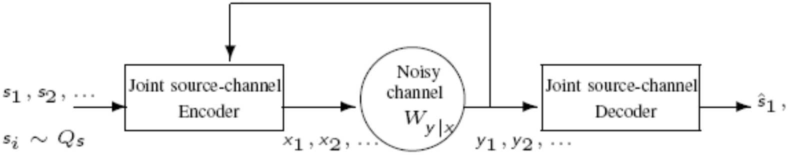

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  - Is feedback worth using?

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   - Review of block joint source-channel coding
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3. Conclusions and Open problems

Review: block coding perspective on channel quality

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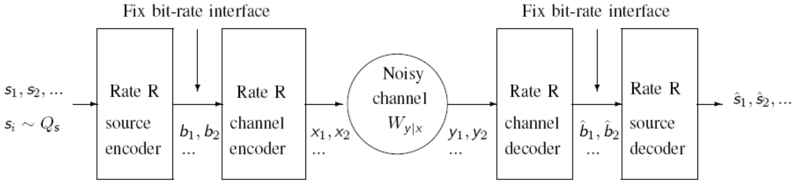

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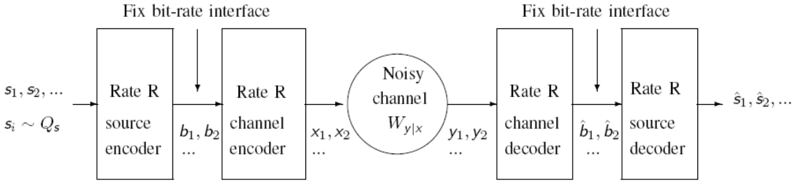

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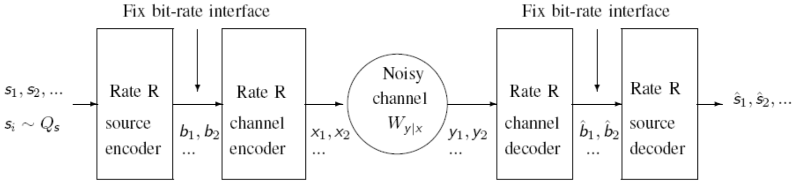

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  - Feedback is irrelevant if channel is memoryless.

Review: How long of a block?

- Decoding error $\Pr(s_1^n \neq \widehat{s}_1^n)$ vs code-length :
  - Error exponent given S, W:

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$$e^b(R) = \min_{P_s : H(P_s) \geq R} D(P_s \| Q_s)$$

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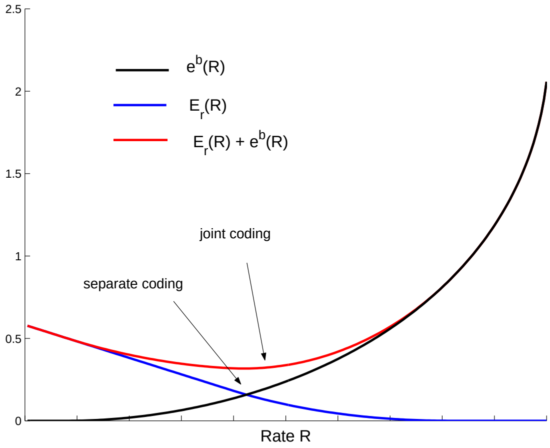
  - Separate source channel coding
    $\Pr(s \neq \widehat{s}) \leq \Pr(b \neq \widehat{b}) + \Pr(s \neq \widehat{s}(b) | b = \widehat{b})$:

$$E^b \geq \max_R \min\{e^b(R), E^b(R)\}$$

The delay price of modularity?

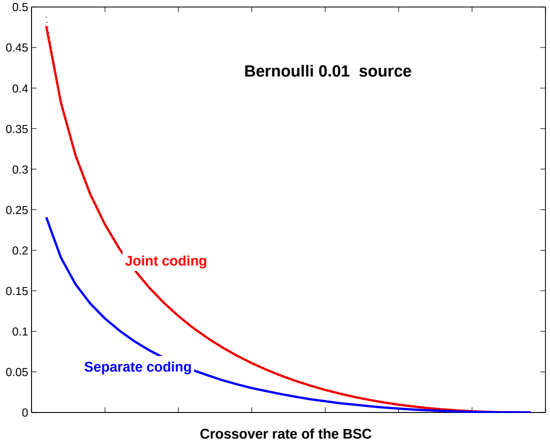
- Joint strictly better than separate (Csiszár):
  - Error exponent (tight if $R^* \geq R_{cr}$):

$$E^b \in [\min_{H(S) \leq R \leq C(W)} \{e^b(R) + E_r(R)\}, \min_{H(S) \leq R \leq C(W)} \{e^b(R) + E_{sp}(R)\}]$$



Pay with delay or with channels

- Random coding error exponents (Bernoulli 0.01 source over BSC)
  - Joint coding: $\min_{H(S) \leq R \leq C(W)} \{e^b(R) + E_r(R)\}$
  - Separate coding: $\max_R \min\{e^b(R), E_r(R)\}$



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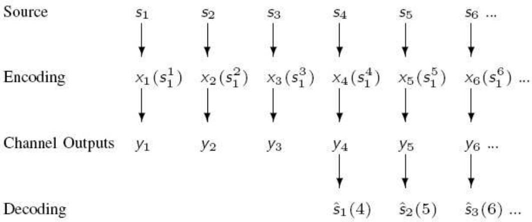
- The real problem with delay constraints:
  - Source symbols $s_1, s_2, ..., s_k, ...$ streaming into the encoder
    Causality: source symbol s_i not available at time $i-1$
  - Finite and fixed system delay



  - Want symbol-wise decoding error tradeoff with delay:

$$\Pr(s_i \neq \widehat{s}_i(i+\Delta)) \sim 2^{-\Delta E^s}$$

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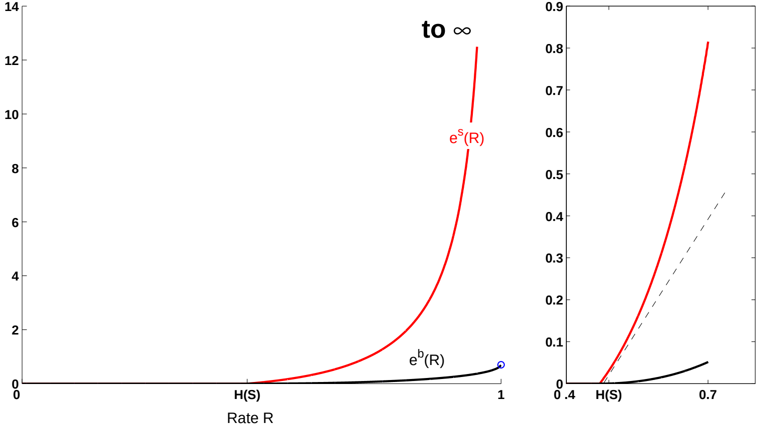
A cause for hope: fixed rate source coding (ITW06)

- Tight bound achieved by variable length coding/queueing:

$$e^s(R) = \min_{P_s,\alpha>0:H(P_s)>(1+\alpha)R} \{\frac{1}{\alpha}D(P_s\|Q_s)\} = \min_{\alpha>0} \frac{1}{\alpha}e^b((1+\alpha)R)$$



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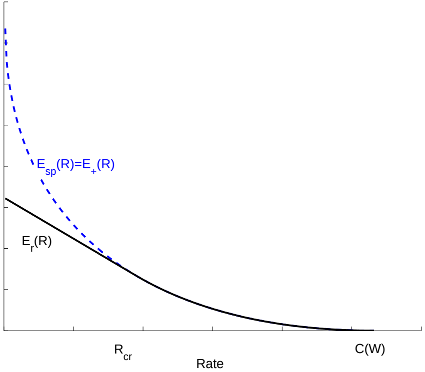
- Upper bounded by Haroutunian bound $E_+(R)$ (Pinsker, Sahai)
  - $E_+(R) = E_{sp}(R)$ for symmetric channels (BEC, BSC...)
  - Proved by feedforward decoding



A joint-coding bound (Allerton 2006)

- Delay constrained lossless source-channel coding error exponent

$$E^s \leq \min_R \{e^s(R) + E_+(R)\}$$

  - $E_+(R) = E_{sp}(R)$ for symmetric channels
  - $e^s(R) = \min_{\alpha>0} \frac{1}{\alpha} e^b((1+\alpha)R)$

- Block coding counterpart (Csiszár):

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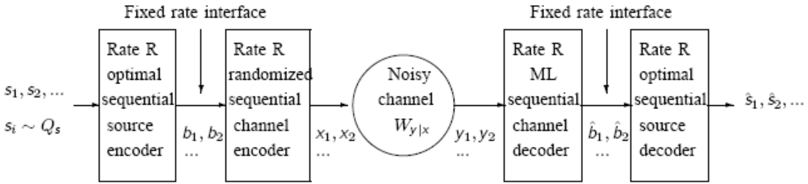
Separate coding (no punctuation)



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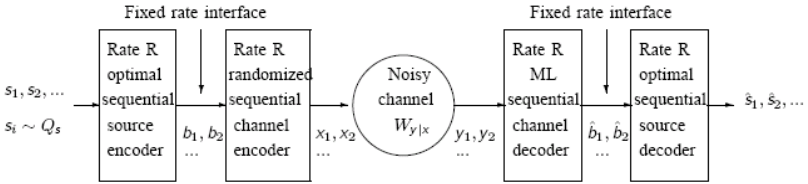

- Union bound on $\Pr(s_i \neq \widehat{s}_i(i+\Delta))$

$$\sum_{k=1}^{i} \Pr(b_k \neq \widehat{b}_k) + \sum_{k=i+1}^{i+\Delta} \Pr(b_k \neq \widehat{b}_k, b_1^{k-1} = \widehat{b}_1^{k-1}) \Pr(s_i \neq \widehat{s}_i(b_1^{k-1}))$$

$$\approx 2^{-\Delta \max_R \min\{e^s(R), E_r(R)\}}$$

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- Drain the source-queue at rate R'.
  - Channel tree branches at variable rate, (t_i for source block i)

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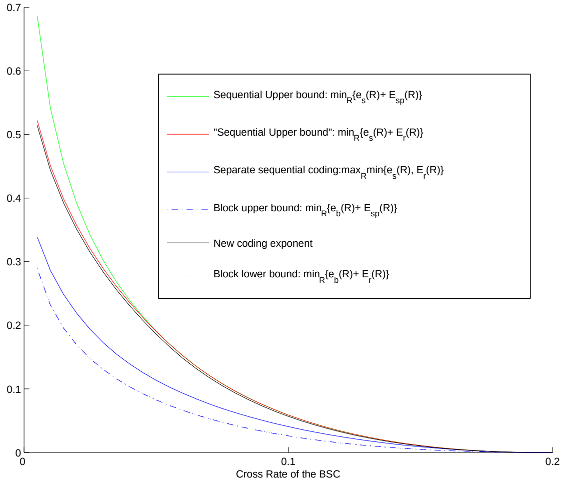
- Error exponent

$$\max_{R'} \min_{\alpha > 0, R \in [0, R']} \{\alpha e^s(R') + e^b((1-\alpha)R) + (1-\alpha)E_r(R)\}$$

Examples (no feedback)

- Delay exponents of a Bernoulli $\frac{1}{20}$ source and BSCs



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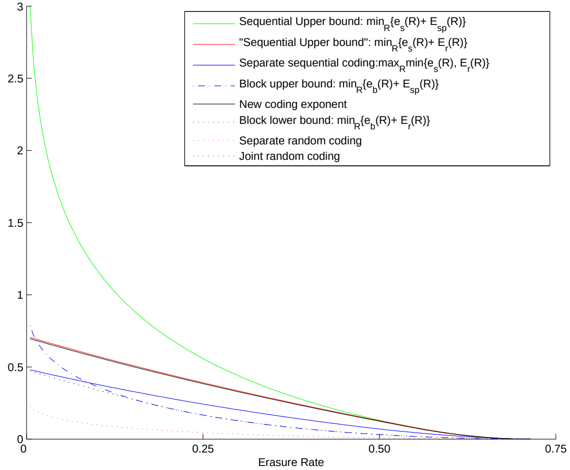

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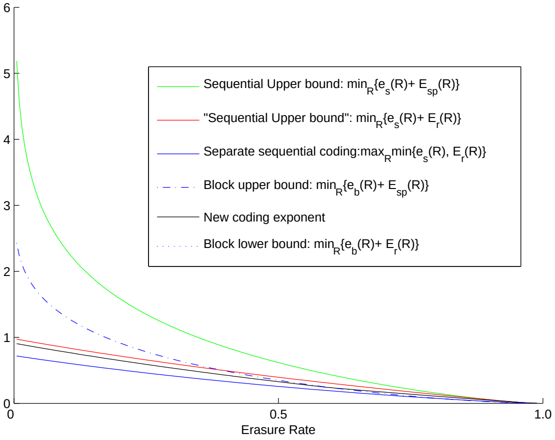

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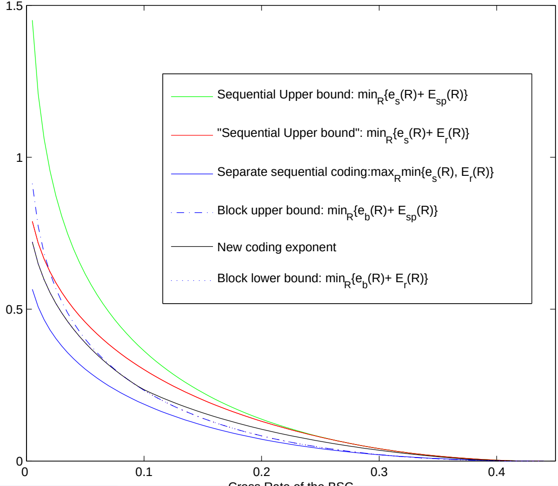

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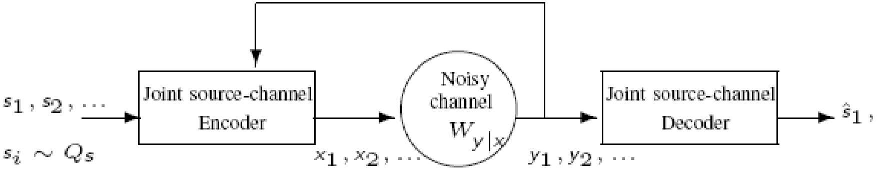

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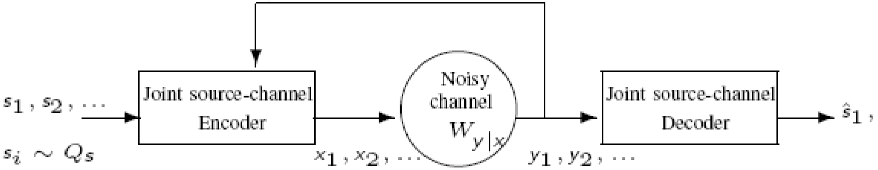
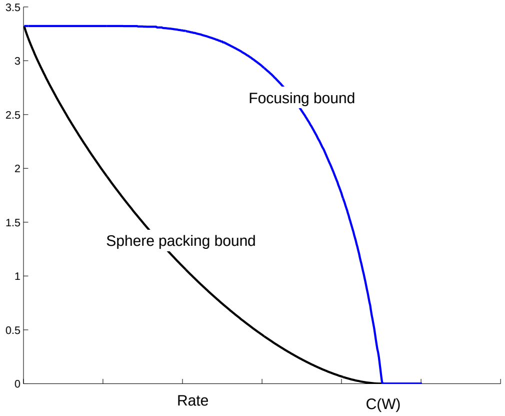


- Capacity unchanged: $H(S) < C(W)$
- What about error exponents?
  - Block coding: upper bound still holds $(\min_R \{e^b(R) + E_+(R)\})$
  - For symmetric channels, upper bound tight if $R^* \geq R_{cr}$
  - No gain with feedback!

More hope: channel coding with feedback (Sahai06)

- Uncertainty focusing bound $\min_{0<\lambda<1} \frac{E_+(\lambda R)}{1-\lambda}$ (Sahai)
  - With "free punctuation," can achieve using variable length codes and queuing



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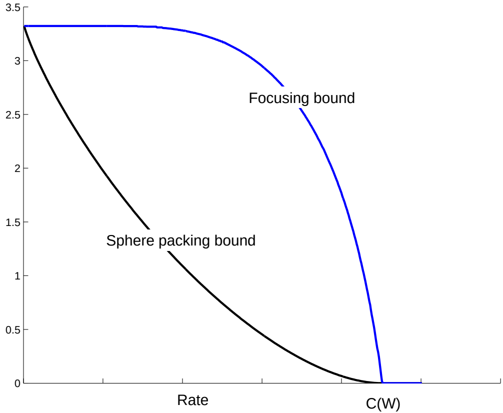

- Can we do better for joint source channel coding with feedback, too?

The upper bound (Allerton 2006)

- Delay constrained lossless source-channel coding error exponent with feedback

$$E_f^s \leq \min_{\lambda \in [0,1], R} \{ \frac{\lambda}{1-\lambda} e(R) + \frac{1}{1-\lambda} E_+(\lambda R) \}$$

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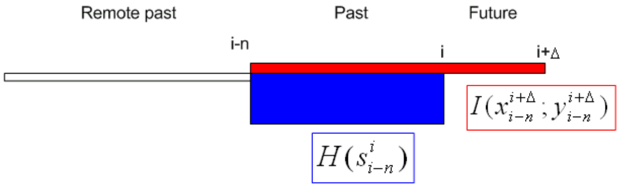
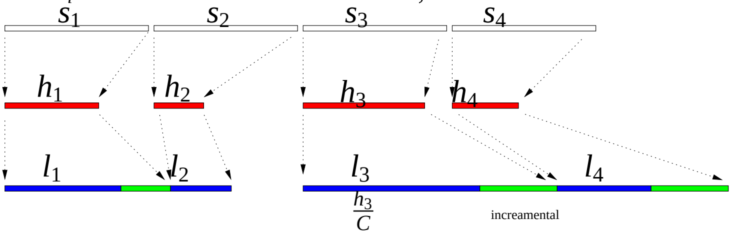
- Proof sketch
  - An symbol error occurs if $H(s_{i-n}^i) > I(x_i^{i+\Delta}; y_i^{i+\Delta})$
  - For a fixed Δ, minimizing over n (dominant error event), $\lambda = \frac{n}{n+\Delta}$



Achieving the bound using free punctuation

- Coding scheme (variable length to variable length)
  - Variable length source coding $s_n \longrightarrow h_n$
  - Increment until success channel coding (variable length)
  - Use $\frac{h_n}{C}$ channel uses then incremental transmission until the decoder can list decode correctly.
  - l_i: channel uses for source block i, i.i.d



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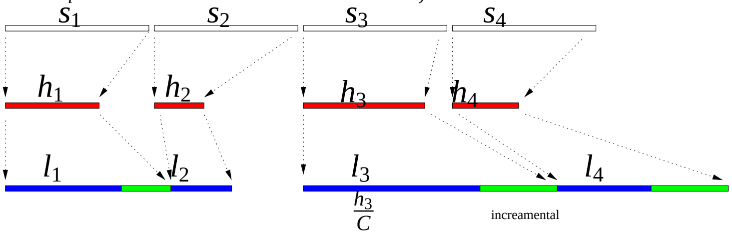

- "Puntuation" is very low rate:
  - Source-block sizes: $\frac{\log n}{n}$
  - Stopping time

How to analyze error probability

- $t_i = \sum_{k=1}^{i} l_i$: time the decoder can decode source symbol i
- With punctuation, s_i can be decoded with 0 error probability at t_i
  - Error event: symbol still in queue.
  - Decoding error for s_i at time $i + \Delta$ iff $t_i > i + \Delta$
  - $\Pr(s_i \neq \widehat{s}_i(i + \Delta)) = \Pr(t_i > i + \Delta)$

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- Apply large deviations analysis:
  $E_f^s = \min_{\lambda \in [0,1], R} \{ \frac{\lambda}{1-\lambda} e(R) + \frac{1}{1-\lambda} E_+(\lambda R) \}$

E_f^s feedback error exponents

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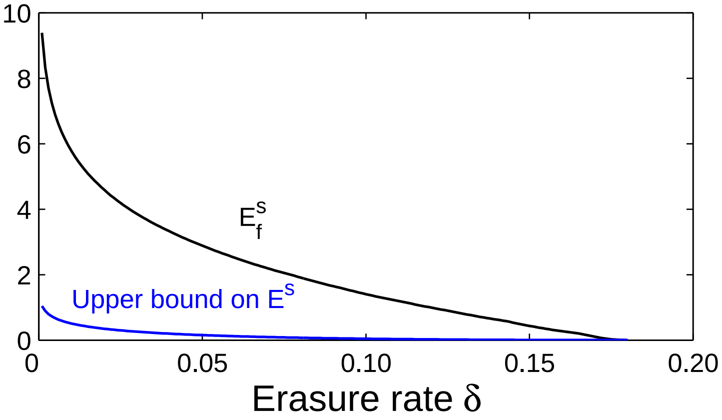

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- Using delay-exponents to understand architecture

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  - Judicious use of "noiseless control channels" seems helpful.

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- Future directions
  - Eliminating or simulating “free punctuation”
  - Truly multiterminal bounding techniques