# "Any-time" Capacity and A Separation Theorem For Tracking Unstable Processes

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*Abstract* — **The problem of tracking an exponentially unstable scalar source process across a noisy channel is considered. We introduce the a new parametric notion of capacity that we call "any-time capacity" $C_{\mathbf{at}}(\alpha)$. It is a twist on the familiar concept of error-exponents and is always between the classical Shannon Capacity and the zero-error capacity. A separation theorem is given which shows that $C_{\mathbf{at}}(\alpha)$ characterizes the properties of a channel needed for finite expected distortion.**

## I. Introduction

In a sense, the justification of Shannon capacity is the classical source-channel separation theorem and its modern refinements[9]. These tell us that for a wide class of sources, channels, and distortion measures, two-part encodings suffice as long as we are willing to tolerate delays.

Traditional rate-distortion theory[1] has focused almost exclusively on stationary processes. While a broad class, it excludes exponentially unstable processes which are important in practice, especially in control applications[5]. Recently, there has been some work showing how to extend source coding to such processes.([7], [3]) But these have implicitly considered only noiseless channels. For noisy channels, the situation was unclear since the traditional source-channel separation theorem need not (and in fact, does not) apply.

## II. Why Classical Separation Fails

Consider the simplest of all unstable processes:

$$X_{t+1} = AX_t + W_t, \quad t \geq 0, A > 1 \tag{1}$$

where $\{X_t\}$ is an $\mathbb{R}$-valued state process and $\{W_t\}$ is a bounded noise process s.t. $\|W_t\| \leq \frac{\omega}{2}$. Assume $X_0 = 0$ for convenience. This process is non-stationary and has infinite variance as $t$ goes to $\infty$. Our per-letter distortion measure is the usual $d(X,\hat{X}) = (X - \hat{X})^2$.

$\forall \delta > 0$, sequential rate distortion theory([8], [7]) gives encoders which can track this process with finite expected distortion using $(\log_2 A + \delta)$ bits per sample. They quantize $(X_t - A\hat{X}_{t-1})$ at each time and recursively track the source.

If we attempt to apply the usual separation results, we would pick an $\epsilon > 0$ which $\exists(N, \mathcal{E}_N, \mathcal{D}_N)$ for which $Pe(\mathcal{E}_N, \mathcal{D}_N) < \epsilon$ across a noisy channel. For Shannon Capacity, while this per-bit probability of error can be made arbitrarily small, it can not be made exactly zero. Eventually, a mistake will be made. The effect will be compounded at every subsequent time step since it will get repeatedly multiplied by $A > 1$ in the source decoder's recursion. The expected per-letter distortion will thus tend to infinity with probability one, regardless of how small an $\epsilon$ we choose in our channel code!

[1]Work under Prof. Sanjoy Mitter and supported by U.S. Army Grant PAAL03-92-G-0115.

## III. "Any-time" Capacity

**Definition III.1** *The* $\alpha$-any-time capacity $C_{at}(\alpha)$ *of a channel is the maximal rate at which the channel can be used to transmit data with a probability of error that decays to zero with delay at least exponentially at a rate* $\alpha$.

$$C_{at}(\alpha) = \sup\{R | \exists(\mathcal{E}^R, K), \forall N > 0, \exists \mathcal{D}_N^R, Pe(\mathcal{E}^R, \mathcal{D}_N^R) < K2^{-\alpha N}$$

The above definition is very close to the definition of the reliability function $E(R)$ of a channel given in [2]. The crucial difference is that while we require the encoder to be fixed, in the standard definition of error exponents both the encoder and decoder vary with delay $N$.

**Theorem III.1** *[6] For the AWGN channel with noiseless feedback,* $C_{at}(\alpha) = C$ *regardless of the value for* $\alpha$.

**Theorem III.2** *[5] For the binary erasure channel with noiseless feedback and probability of erasure* $e$*:*

$$C_{at}(\eta - \log_2(1 + (2^\eta - 1)e)) = 1 - \frac{1}{\eta}\log_2(1 + (2^\eta - 1)e)$$

*if you let* $\eta$ *range over* $(0, \infty)$.

Amazingly, [6] shows that $\alpha$-any-time capacity is also nonzero for these channels even without any feedback!

## IV. Separation For Unstable Processes

**Theorem IV.1** *The source in (1) can be tracked with finite MSE across a noisy channel iff there is an* $\epsilon > 0$ *for which* $C_{at}(2\log_2 A + \epsilon) > \log_2 A$ *for the channel.*

## References

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