Evaluating Channels for Control: Capacity Reconsidered

ACC00-INV2305

Anant Sahai [1]

Department of Electrical Engineering and Computer Science

and Laboratory for Information and Decision Systems

Massachusetts Institute of Technology

Cambridge, Massachusetts, 02139

sahai@lids.mit.edu

Abstract

We show that Shannon's classical notion of capacity is not enough to characterize a communication channel if we intend to use that channel as a part of a feedback loop. This is done by considering the stabilization problem for a simple discrete time linear system. While classical capacity is not enough, we show by example that the stricter notion of Shannon's zero-error capacity is conservative. We finish by introducing a new parametric notion of capacity that we call "any-time capacity" and evaluate it for two channels with feedback: the continuous valued additive white Gaussian noise channel and the binary erasure channel.

1 Introduction

For communication theorists, Shannon's classical channel capacity theorems are not just beautiful mathematical results, they are useful in practice as well. They let us summarize a diverse range of channels by a single figure of merit: the capacity. For most non-interactive communication applications, the Shannon capacity of a channel is the true upper bound for performance.

It is natural to ask whether Shannon's classical capacity is also the right characterization for channels being used in control systems. Recent work on sequential rate distortion theory has shown that there is a natural notion of rate that can be attached to an unstable linear discrete-time process.[9] This is related to work in estimation being carried out by Nair and others using noiseless digital channels.[2] We had also previously showed that it is possible to stabilize controlled Gauss-Markov processes over power-constrained additive white Gaussian noise channels.[5] However, while AWGN channels are useful models for many physical situations, they are not relevant to noisy digital networks like the Internet where packets are sometimes lost.

[1]This research supported by U.S. Army Grant PAAL03-92-G-0115, Center for Intelligent Control Systems.

After reviewing some definitions and basic problem setup in the sections 2 and 3, we will consider controlling a simple unstable scalar plant over a binary erasure channel in section 4. We show that while it is possible to stabilize the system, having classical Shannon capacity simply greater than the intrinsic rate is not enough! At the same time, we show that Shannon's zero error capacity is too conservative since it is always zero for this channel: so neither one is the right capacity concept.

In section 5, we propose a new parametric notion of channel capacity called "α-any-time capacity."[6] This notion is always between Shannon's classical capacity and zero-error capacity. We claim that *any-time capacity* is the relevant characterization of a noisy channel for the purposes of feedback control.

2 Channels and Classical Notions of Capacity

Definition 2.1 *A* discrete time memoryless channel *is a probalistic system with an input. At every time step t, it takes an input $a_t \in \mathcal{A}$ and produces an output $b_t \in \mathcal{B}$ with probability $p(b_t|a_t)$. Conditioned on a_t, b_t is independent of any other random variable in the system.*

Definition 2.2 *The* Shannon classical capacity C *of a channel is the maximal rate at which the channel can be used to transmit data with an arbitrarily small probability of error.[1]* $C = \sup\{R|\forall\epsilon > 0\exists N, \mathcal{E}_N^R, \mathcal{D}_N^R P_{error}(\mathcal{E}_N^R, \mathcal{D}_N^R) < \epsilon\}$

Definition 2.3 *The* Shannon zero-error capacity C *of a channel is the maximal rate at which the channel can be used to transmit data without error.[8]* $C_0 = \sup\{R|\exists N, \mathcal{E}_N^R, \mathcal{D}_N^R P_{error}(\mathcal{E}_N^R, \mathcal{D}_N^R) = 0\}$

In both definitions of capacity, the encoder/decoder pair $\mathcal{E}_N^R, \mathcal{D}_N^R$ has rate R and end-to-end delay less than or equal to N. By inspection, we know $C_0 \leq C$.

Definition 2.4 *The* binary erasure channel *has* $\mathcal{A} = \{0,1\}$, $\mathcal{B} = \{0,1,\emptyset\}$ *and* $p(0|0) = p(1|1) = 1-e$ *while* $p(\emptyset|0) = p(\emptyset|1) = e$.

It can be shown that the Shannon classical capacity of the erasure channel is $1-e$ bits per channel use regardless of whether the encoder has feedback.[1] Furthermore, because a long string of erasures is always possible, the Shannon zero-error capacity of this channel is 0 as long as $e > 0$.

Definition 2.5 *An* AWGN channel *is an analog channel modeled as* $b_k = a_k + v_k$ *where* $\mathcal{A} = \mathcal{B} = \mathbb{R}$, *and* $\{v_k\}$ *is an IID Gaussian process with zero mean and variance* K_V *representing the channel noise. Furthermore, to prevent degenerate solutions, we impose the following power constraint:* $E(\|a_k\|^2) < P$ *for some total power* P *per time step.*

It can be shown that the Shannon classical capacity of the AWGN channel is $\frac{1}{2}\log_2(1+\frac{P}{K_V})$ bits per channel use regardless of whether the encoder has feedback.[1] Traditionally, zero-error capacity has not been evaluated for continuous channels. But the same logic applies as before and a string of rare events is always possible, so the Shannon zero-error capacity of this channel is 0 as long as $K_V > 0$.

3 The Simple Control System

We consider the following scalar control system:

$$X_{t+1} = AX_t + U_t + W_t, \quad k \geq 0 \tag{1}$$

where $\{X_t\}$ is an $\mathbb{R}$-valued state process. $\{U_t\}$ is an $\mathbb{R}$-valued control process and $\{W_t\}$ is a bounded noise/disturbance process s.t. $||W_t|| \leq \frac{\omega}{2}$. $A > 1$ so the system is unstable.

We also introduce a distributed nature to the problem by placing a communication channel in the feedback loop. We require an observer/encoder system $\mathcal{E}$ to observe X_t and generate inputs a_t to the channel. We also require a decoder/controller system $\mathcal{D}$ to observe channel outputs b_t and generate control signals U_t. We allow both $\mathcal{E}, \mathcal{D}$ to have arbitrary memory.

For general linear systems, the intrinsic minimal rate of the system is the sum of the $\log_2$ of the unstable eigenvalues.[11] This means that it is certainly impossible to stabilize our scalar system if the feedback channel's Shannon classical capacity $C < \log_2 A$. We also have explicit schemes which can stabilize such systems over noiseless digital channels with $C_0 = C > \log_2 A$.[11] In addition, if the disturbance process is stochastic i.i.d. with bounded mean and variance, we can also stabilize it in the natural stochastic sense over AWGN channels with $C > \log_2 A$.[5] So, what happens when we consider the binary erasure channel? Is $C = 1 - e > \log_2 A$ enough?

4 Control Over The Erasure Channel

The first thing is to specify the notion of 'stability'. Since the channel is stochastic, it is natural to want to use the natural stochastic definition of keeping the variance of X_t finite. However, this approach runs up against certain very difficult and unsolved problems in measure theory. So, we take a different tack here and resist making any assumptions on the stochastic nature of the disturbance. Instead, we consider this as a "mixed" system in which all we know about the disturbance is that it is bounded.

Now, consider an additional external observer located at the decoder.[4] This observer knows the encoder and the control law being used, but can only see the outputs of the channel. Suppose that before seeing b_t the observer knew that $X_t \in [l, u]$. Now, imagine that an erasure occurs and $b_t = \emptyset$. The observer has received no information and can thus only update the uncertainty to conclude $X_{t+1} \in [Al - \frac{\omega}{2} + U_{t+1}, Au + \frac{\omega}{2} + U_{t+1}]$. If, on the other hand $b_t \neq \emptyset$, then this will enable the observer to throw away some part of the uncertainty interval. Regardless, the uncertainty remains an interval and has a well defined length. This suggests the following natural notion of stability:

Definition 4.1 *Let Y_t represent the size of the external observer's uncertainty about the state of the system. A mixed system is* stable *if* $\exists M, \forall t, \forall \{W_t\}, E(Y_t^2) \leq M$

4.1 Necessity: What do we need?

Since the disturbances are unknown, the best any potential encoding/decoding system can do is cut the uncertainty in half for every bit correctly received by the decoder. The optimal Y_t process is: $Y_{t+1} = AY_t + \omega$ if an erasure occurs and $Y_{t+1} = \frac{A}{2}Y_t + \omega$ otherwise. Assume $Y_0 = 0$.

The Y_t are now a well-defined stochastic process since the channel is assumed to be memoryless. At every time step, a new ω is added in while all the previous ω's get multiplied by a random factor. Write Y_t out as a sum: $Y_t = \sum_{n=0}^{t-1} \omega A^n F_n$ where the F_n are correlated random variables which express how many of the last n channel transmissions have gotten through. In particular $F_n = \prod_{k=t-n}^{t} (\frac{1}{2})^{I_k}$ where I_k is the indicator for the event that the k-th transmission was not erased. Individually, the probability distribution of the F_n distributions is easy to see: $F_n = \frac{1}{2^i}$ with probability $\frac{n!(1-e)^i e^{n-i}}{i!(n-i)!}$.

We can now consider the asymptotic properties of Y_t^2

by formally considering:

$$E(Y_\infty^2) = E(\omega^2 \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} A^{j+n} F_j F_n)$$

Define N_k^j as the random variable for which $F_{j+k} = F_j N_k^j$. Then group terms:

$$\begin{aligned} E(Y_\infty^2) &= E(\omega^2 \sum_{j=0}^{\infty} A^{2j} F_j^2 (1 + 2\sum_{k=1}^{\infty} A^k N_k^j)) \\ &= \omega^2 \sum_{j=0}^{\infty} A^{2j} (E(F_j^2) + 2\sum_{k=1}^{\infty} A^k E(F_j^2 N_k^j)) \end{aligned}$$

F_j and N_k^j are independent by construction, and hence expectations distribute over them. Moreover, by the memorylessness of the channel, it is clear that $E(N_k^j)$ does not depend on j at all, and in fact $E(N_k^j) = E(F_j)$. So we get: $\frac{E(Y_\infty^2)}{\omega^2}$

$$\begin{aligned} &= \left(\sum_{j=0}^{\infty} A^{2j} E(F_j^2)\right)\left(1 + 2\sum_{k=1}^{\infty} A^k E(F_k)\right) \\ &= \left(\sum_{j=0}^{\infty} A^{2j} \sum_{i=0}^{j} \frac{j!(1-e)^i e^{j-i}}{i!(j-i)!4^i}\right) \\ &\quad \left(1 + 2\sum_{k=1}^{\infty} A^k \sum_{l=0}^{k} \frac{k!(1-e)^l e^{k-l}}{l!(k-l)!2^l}\right) \\ &= \left(1 + \sum_{j=1}^{\infty} (A^2)^j e^j \sum_{i=0}^{j} \frac{j!}{i!(j-i)!}(\frac{1-e}{4e})^i\right) \\ &\quad \left(1 + 2\sum_{k=1}^{\infty} A^k e^k \sum_{l=0}^{k} \frac{k!}{l!(k-l)!}(\frac{1-e}{2e})^i\right) \\ &= \left(1 + \frac{A^2(1+3e)}{4 - A^2(1+3e)}\right)\left(1 + \frac{2A(1+e)}{2 - A(1+e)}\right) \end{aligned}$$

More importantly, this is only valid if $A < \frac{2}{\sqrt{1+3e}} < \frac{2}{1+e} < 2^{1-e}$. Otherwise, the sums diverge. This shows that $\log_2 A < 1 - e$ is not sufficient and so, the Shannon classical capacity is not the right way to evaluate channels in a feedback loop.

4.2 Sufficiency: Achieving the new bound

The above analysis implicitly required the observer/encoder to always be exactly synchronized with the decoder/controller. This need for signaling is intrinsic in most general distributed control problems.[3] In our case, we can use the plant itself to signal noiselessly from the decoder to the encoder.

Assume that $\mathcal{E}$ and $\mathcal{D}$ start out perfectly in sync. The encoder transmits a 0 if the X_t is in the bottom-half of the uncertainty region and a 1 otherwise. If the decoder receives the bit, it applies whatever control value is needed to center the (now halved) uncertainty region around the origin. If, the bit is erased, the decoder uses a special large value of U_{signal} which is greater than 2ω. Since the encoder knows the last value of X_t exactly, it can detect this signal and conclude that the decoder's uncertainty region has not shrunk. At the next time step, the decoder will use a control value that counteracts the previously introduced communication by subtracting AU_{signal} from whatever control value it would have used otherwise.

After picking U_{signal}, the above description uniquely specifies a scheme which will always stabilize the system if $A < \frac{2}{\sqrt{1+3e}}$. Thus the bound we have is both necessary and sufficient.

5 "Any-time" Capacity

So Shannon's classical capacity is too loose, and zero-error capacity is too conservative. Is there anything in between that works?

Definition 5.1 $C_{anytime}(\alpha)$, *the* α-any-time capacity, *is the maximum rate at which the channel can be used to transmit data with a probability of error that decays at least exponentially with delay at a rate* α. $C_{anytime}(\alpha) = \sup\{R | \exists \mathcal{E}^R, K > 0 \forall N \exists \mathcal{D}_N^R P error(\mathcal{E}^R, \mathcal{D}_N^R) < K 2^{-\alpha N}\}$

The above definition is close to the definition of the reliability function or error exponent $E(R)$ of a channel given in [1]. But In the standard definition of $E(R)$, we let both the encoder and decoder vary with delay N. In our definition of α-any-time capacity, we require the encoder to be fixed.

This is why we call it "any-time" capacity. We can stop the decoding process for a given bit at "any-time" and require the answer to be increasingly meaningful. The α specifies the rate at which we want the answers to improve. It should be clear that $\forall\alpha, C_0 \leq C_{\text{anytime}}(\alpha) \leq C$. It turns out that:

Theorem 5.2 *An unstable scalar linear system with intrinsic rate* $\log_2 A$ *is stabilizable over a noisy channel iff* $C_{anytime}(2\log_2 A) > \log_2 A$ *for the channel with feedback.*

We omit the proof here because it is quite involved and needs some special constructions. By setting up a related estimation problem over a channel with feedback[6], we can prove the forward half of the theorem and show that "any-time capacity" is a sufficient condition for stabilization. The necessity is much harder to show and requires some new techniques.[7]

Furthermore, these same constructions can be combined with the controller used in [5] to show the following interesting result which shows us why we did not see any gap for the AWGN channel:

Theorem 5.3 *For the AWGN channel with feedback,* $\forall\alpha > 0, C_{anytime}(\alpha) = C$

For the binary erasure channel, we conjecture the following implicit formula.

Theorem 5.4 *For the binary erasure channel with feedback, let* η *range over* $(0, \infty)$*:*

$$C_{anytime}(\eta - \log_2(1 + (2^\eta - 1)e)) = 1 - \frac{1}{\eta}\log_2(1 + (2^\eta - 1)e)$$

This has the right asymptotes: $C_{\text{anytime}}(0) = 1 - e$ and $C_{\text{anytime}}(\alpha \geq -\log_2 e) = 0$.

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