Characterization and Computation of Local Nash Equilibria in Continuous Games

Lillian J. Ratliff, Samuel A. Burden, S. Shankar Sastry
Electrical Engineering and Computer Sciences
University of California, Berkeley

Abstract

We present derivative-based necessary and sufficient conditions ensuring player strategies constitute local Nash equilibria in a non-cooperative continuous games. Our results can be interpreted as generalizations of analogous first- and second-order conditions for local optimality from nonlinear programming and optimal control theory. Drawing on this analogy, we propose an iterative steepest descent algorithm for numerical approximation of local Nash equilibria and provide a sufficient condition ensuring local convergence of the algorithm. We demonstrate our analytical and computational techniques by computing local Nash equilibria in games played on a finite-dimensional differentiable manifold or an infinite-dimensional Hilbert space.

Parallelizing Non-Linear Programming and Optimal Control

Motivated by systems with myopic agents and non-convex strategy spaces, through parallelizing non-linear programming and optimal control we seek a characterization for local Nash equilibria that is computable.

Single cost optimization vs. Competitive Optimization

Characterization and Computation

- Generalizing derivative-based conditions for local optimality from nonlinear programming and optimal control, we derived necessary first- and second-order conditions that local Nash equilibria must satisfy.
- Using a dynamical systems perspective, we develop an algorithm for computation.
- By combining these analytical and computational advancements, we anticipate developing novel schemes for decentralized control in engineered systems as well as new identification techniques for human agents.

Game Formulation

- Two rational, selfish agents, Urbain and Victor, that behave myopically and have topological spaces $M_1$, $M_2$ respectively as strategy spaces.
- Urbain’s cost function is $J_1: M_1 \times M_2 \to \mathbb{R}$ and Victor’s is $J_2: M_1 \times M_2 \to \mathbb{R}$.
- A strategy $(p, q) \in M_1 \times M_2$ is a local Nash equilibrium if there exist open sets $W_1 \subset M_1$, $W_2 \subset M_2$ such that $p \in W_1$, $q \in W_2$, and the following holds:
  $J_1(p, q) \leq J_1(p', q), \forall p' \in W_1 \setminus \{p\}$
  $J_2(p, q) \leq J_2(p, q'), \forall q' \in W_2 \setminus \{q\}$.

Finite-Dimensional Strategy Spaces

- Let $M_1$, $M_2$ be smooth manifolds without boundary. A differential game form is a differential $1$-form $\omega: M_1 \times M_2 \to \mathbb{T}(M_1 \times M_2)$ defined in local coordinates by $D_1 J_i(p, q, D_2 J_i(p, q))$.
- $\omega$ indicates the direction in which Urbain and Victor can change their strategies to decrease their individual cost functions most rapidly.
- Suppose that $M_1$ and $M_2$ are finite-dimensional smooth manifolds. Then, a strategy $(p, q) \in M_1 \times M_2$ is a differential Nash equilibrium if $\omega(p, q) = 0$ and $D^2_{pq} J_i(p, q) > 0$. Theorem

Proposition

A local Nash equilibrium $(p, q) \in M_1 \times M_2$ satisfies $D_i J_i(p, q) = 0$ and $D^2_{pq} J_i(p, q) \geq 0$ for each $i \in \{1, 2\}$.

Infinite-Dimensional Strategy Spaces

- $M_i$ is the space of square integrable and bounded functions on $[0, T]$, $i = 1, 2$.
- $\mu_i \in M_i$ is the $i$-th player’s strategy choice. $x(t) \in \mathbb{R}^n$ is the state of the game with dynamics $\dot{x}(t) = f(x(t), \mu_1(t), \mu_2(t))$. $\forall t \in [0, T]$.
- $x^{(\mu_1, \mu_2)}(T)$ is the state at time $T$ given $x_0$, $\mu_1$, and $\mu_2$.
- Each $J_i(x^{(\mu_1, \mu_2)}(T))$ is $C^2$ Fréchet-differentiable.
- We pose each player’s optimization problem as
  $\min_{\mu \in M_i} J_i(x^{(\mu_1, \mu_2)}(T))$.

Theorem

Suppose $(\mu_1^*, \mu_2^*)$ is a local Nash equilibrium. Then, $D_1 J_i \equiv 0$ and $D^2_{pq} J_i(\mu_1^*, \mu_2^*)$ is a positive semi-definite bilinear form on $W_i \subset M_i$, for each $i \in \{1, 2\}$. Conversely, if $(\mu_1^*, \mu_2^*)$ a differential Nash equilibrium, i.e. $D_1 J_i \equiv 0$ and $D^2_{pq} J_i(\mu_1^*, \mu_2^*)$ is a positive definite bilinear form, then it is a strict local Nash equilibrium.

Decoupled, Myopic Approximate Best Response

- Consider a two-player game over the finite-dimensional strategy space $U_1 \times U_2$ with player costs $J_1, J_2: U_1 \times U_2 \to \mathbb{R}$.
- We study the continuous-time dynamical system generated by the negative of the gradients $D_i J_i$.
- With $u = (u_1, u_2) \in U_1 \times U_2$, we let
  $\dot{u}_1 = \frac{\partial}{\partial u_1} J_1(u_1, u_2)$
  $\dot{u}_2 = \frac{\partial}{\partial u_2} J_2(u_1, u_2) = -\omega(u_1, u_2)$.
- Forward-Euler approximation of dynamical system for $u$:
  $u^{n+1} = u^n - \frac{\omega(u^n)}{\eta}$ (FE)

A differential Nash $\mu$ is a fixed point of (FE).

- Linearizing about $\mu$, if all eigenvalues of $-D\omega(\mu) \in \text{OLHP}$, then $\exists \eta > 0$ s.t. $\forall h \in (0, \eta), \mu$ is an exponentially stable fixed point of (FE).
- This ensures local convergence of iterates of (FE).
- Algorithm: Iterate (FE) until $\|\omega(u^n)\|$ is sufficiently small.

Location Game

Two-player game on the unit circle:

$J_1(\theta_1, \theta_2) = -\cos \theta_1 + \alpha_1 \cos(\theta_1 - \theta_2)$

$J_2(\theta_1, \theta_2) = -\cos \theta_2 + \alpha_2 \cos(\theta_2 - \theta_1)$

- Our theorem implies any point $(\theta_1, \theta_2)$ for which $\omega(\theta_1, \theta_2) = 0$ and $D\omega(\theta_1, \theta_2) \geq 0$ is an isolated local Nash equilibrium.
- There are two differential Nash equilibria situated symmetrically around the zero angle: one near $(1, -1.1)$ and the other near $(-1, 1.1)$.
- The blue and green regions are empirical basins of attraction for the differential Nash equilibrium located at the blue circle and the green circle respectively.
- Points where $\omega = 0$ but $D\omega$ is not positive-definite are located at $(0, \pi)$ and $(\pi, \pi)$.
- As we vary $\alpha_2$, the equilibria collapse on one another. This motivates a bifurcation analysis.

References