Iterated Strict Dominance, Rationalizabilty, and Correlated Equilibrium

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Motivation

- How to find NE?
  - Player will not play strategies that are strictly dominated: they can be (iteratively) deleted $\Rightarrow$ISD
  - Player will not play strategies that are not best response to some strategy of the opponents: keep only those best responses $\Rightarrow$ Rationalizability
- In NE players are assumed to choose strategies independently.
  - What happens if the choices are correlated? Correlated equilibrium

Iterated Strict Dominance

- Successively eliminate strictly dominated strategies

	L	R
U	1,3	4,1
D	0,2	3,4

	L	R
U	1,3	4,1

	L
U	1,3

- Lead to the NE for this example.

Iterated Strict Dominance

Definition 1. *Set* $S_i^0 = S_i$ *and* $\Sigma_i^0 = \Sigma_i$ *and define* S_i^n *recursively by*
$S_i^n = \{s_i \in S_i^{n-1} | \nexists \sigma_i \in \Sigma_i^{n-1}$ *such that* $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}^{n-1}\}$
and define
$\Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0$ *only if* $s_i \in S_i^n\}$*.*
Set $S_i^\infty = \bigcap_{n=0}^\infty S_i^n$ *and*
$\Sigma_i^\infty = \{\sigma_i \in S_i^\infty | \nexists \sigma_i'$ *such that* $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}) \forall s_{-i} \in S_{-i}^\infty\}$

Iterated Strict Dominance

- Σ_i^∞ may be smaller than the set of mixed strategies over S_i^∞. Exple

	L	R
U	1,3	-2,0
M	0,-2	1,3
D	0,1	0,1

  $S_i^\infty = S_i$ however $\frac{1}{2}(U, M)$ is dominated by D and does not belong to Σ_i^∞

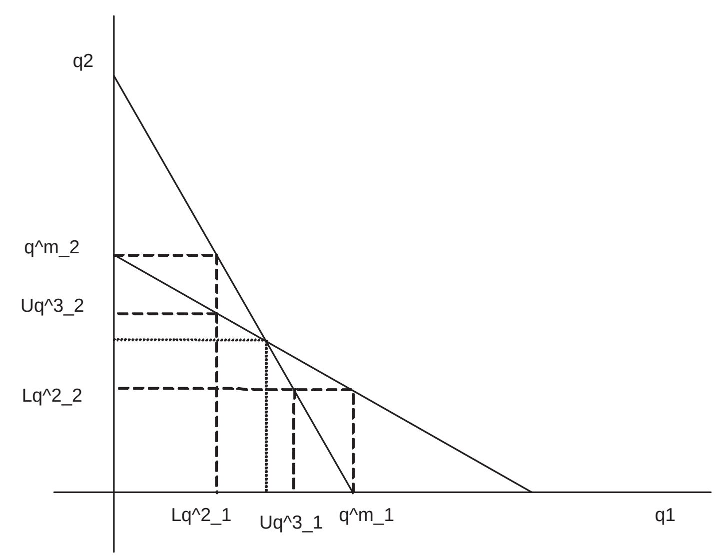
- In finite game iteration cease to delete after some n

ISD: Remarks

- We have assumed that at each iteration all dominated strategies of all players are deleted simultaneously
- We could as well delete for player one first, then for player two, and so on. The limit $S^\infty = S_i^\infty \times \cdots \times S_I^\infty$ does not depends on the deletion order (not the case for weakly dominance)
- At each round we only delete pure strategies: the limit Σ_i^∞ would be the same if we deleted pure and mixed strategies at each round.
- We say that a game is solvable by ISD if S_i^∞ is a singleton

Example Cournot Model

- Two firms choose output levels $q_i \in Q_i = [0, \infty)$.
- They sell at market clearing price $p(q)$ where $q = q_1 + q_2$.
- The utilities are $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$ where $c_i(q_i)$ is the cost of output level q_i.
- Assume: u_i are strictly concave, reaction curves (best response for each output of the other firm) intersect only in one point and are decreasing.



Rationalizability

- ISD is based on the principle that: players should not play strategies that are strictly dominated.
- Contrapositive: players should not plat strategies that are not best response to some beliefs about the opponents' strategies.
- Such strategies can be iteratively deleted: *rationalizability*.

Rationalizability

Definition 2. *Rationalizabilty is defined by the following iterative process: Set* $\tilde{\Sigma}_i^0 = \Sigma_i$*, and for each* i *recursively define*

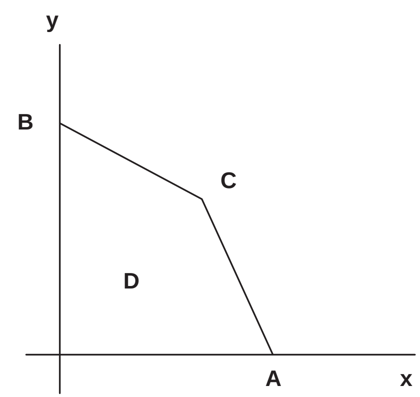
$$\tilde{\Sigma}_i^n = \Big\{\sigma_i \in \tilde{\Sigma}_i^{n-1} \big| \exists \sigma_{-i} \in \times_{j \neq i} \text{ convex hull } (\tilde{\Sigma}_i^{n-1}) \text{ such that } u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \forall \sigma_i' \in \tilde{\Sigma}_i^{n-1} \Big\}$$

The rationalizable strategies for player i *is* $R_i = \bigcap_{n=0}^{\infty} \tilde{\Sigma}_i^n$*.*
A strategy σ *is* rationalizable *if* σ_i *is rationalizable for each player* i*.*

Rationalizability

- Why the convex hull? Even though $\sigma, \sigma' \in \tilde{\Sigma}_i^n$, mixtures might not be.

S1	x	y
A	3	0
B	0	3
C	2	2
D	1	1



- Every NE is rationalizable: a NE σ^* is never dominated
- Theorem(Bernheim): The set of rationalizable strategies is never empty and contains at least one pure strategy for each player

Rationalizability and ISD

- The set of rationalizable strategies is contained in the set that survives ISD.
  - A strictly dominated strategy is never a best response
  - ISD delete strategies that are strictly dominated
  - Rationalizability delete strategies that are not best responses
- The converse is true in a two-player game (Intuition?) (Theorem by Pearce: proved using induction and separating hyperplane theorem)
- Does not hold in games with three or more players: set of mixed strategies is not convex?
  However it holds for *correlated mixed strategies*.

Rationalizability and ISD: example

- P1 plays row, P2 column, P3 matrix; they all have the same payoff
- 3^{rd} matrix is **not** strictly dominated for P3: $(1/2, 1/2, 0, 0)$ has the same payoff
- However it is never a best response for any belief about the other players

If P1 $(x, 1-x)$, P2 $(y, 1-y)$ then $M_1 \Rightarrow 8xy$, $M_2 \Rightarrow 8(1-x)(1-y)$, $M_3 \Rightarrow 8xy + 4(1-x-y)$, $M_1 \Rightarrow 3$
If $M_3 \geq (M_1, M_2)$ then $x + y = 1$ and $3 > 8xy$

	L	R
U	8	0
D	0	0

	L	R
U	0	0
D	0	8

	L	R
U	4	0
D	0	4

	L	R
U	3	3
D	3	3

Correlated Equilibrium

Example1

	L	R
U	5,1	0,0
D	4,4	1,5

- Three equilibria: pure $(U, L), (D, R)$, and mixed $(1/2, 1/2); (1/2, 1/2)$ that gives 2.5 to each player.
- Now suppose that players observe $X \sim Unif\{A, B, C\}$ (**correlation device**), $P1$ ($P2$) is perfectly informed if A (C) occurs and otherwise does not know if B or C (A or B) occurred.
- Consider the following strategy: $P1$ ($P2$) plays U (R) when told A (C) and D (L) otherwise.
- This is a NE with outcomes $(U, L), (D, L), (D, R)$ each with proba $1/3$ (never (U, R)) and the expected payoffs are $3\frac{1}{3}$

Correlated Equilibrium

Example2

	L	R
U	0,1,3	0,0,0
D	1,1,1	1,0,0

A

	L	R
U	2,2,2	0,0,0
D	2,2,0	2,2,2

B

	L	R
U	0,1,0	0,0,0
D	1,1,0	1,0,3

C

- P1 chooses row, P2 choose column, P3 chooses matrix
- Unique NE (D, L, A) with payoff (1,1,1)
- Suppose now that P1 and P2 observe a fair coin, P3 does not observe
- A NE for this game is: P1 plays U if H and D if T
  P2 plays L if H and R if T
  P3 plays B

Correlated Equilibrium: definition

- What are the strategies for the *expanded game* with *correlating device*
- Correlating device is a triple $(\Omega, \{H_i\}, p)$, where Ω state space of outcomes of the device, p a probability distribution on Ω
- H_i (information partition): player i's information about which ω occurred (in this case Pi is told that $\omega \in h_i(\omega)$)
- $h_i(\omega)$ is the set of those states that Pi regards as possible when the truth is ω
- Players condition their play on the signal sent by the correlating device.

Correlated Equilibrium: definition

- Pure strategy for the expanded game ξ_i maps elements of Ω to pure strategies $s_i \in S_i$ of the original game such that $\xi_i(\omega) = \xi_i(\omega')$ if $\omega' \in h_i(\omega)$. (Mixed strategies are defined as mixed of these pure strategies)

- **Definition 3.** *A* correlated equilibrium ξ *relative to the correlating device* $(\Omega, \{H_i\}, p)$ *is a NE in strategies adapted to that device. That is* $\xi = (\xi_1, \ldots, \xi_I)$ *is a CE if for all* i *and every strategy* $\tilde{x}i_i$

$$\sum_{\omega \in \Omega} p(\omega) u_i(\xi_i(\omega), \xi_{-i}(w)) \geq \sum_{\omega \in \Omega} p(\omega) u_i(\tilde{\xi}_i(\omega), \xi_{-i}(w))$$

Correlated Equilibrium in Wireless Comm

- Altman, Bonneau, Debbah (INRIA Sophia Antipolis)
- Wireless access channel: slotted Aloha
- Non-cooperative: nodes are selfish
- Cooperative: nodes observe a common/correlated randomness (signaling device)

Model

- m users, subset $Z(t)$ of them is active at each time t with probability $\zeta(Z(t))$, active mobile always have a packet to send
- Strategies: probability of transmission q_i in non-cooperative game, and (p_i, q_i) in cooperative game.
- Power constraint: $q_i \leq q_i^{max}$
- Probability of success at any time slot

$$\Theta_{all}(q_1, \ldots, q_m) = E_z\left[\sum_{i \in z} q_i \prod_{j \in z-i} (1 - q_i)\right]$$

Average throughput Θ_{all}/m

- Throughput of mobile i condition on being active

$$\Theta_i^{act}(q_1, \ldots, q_m) = E_z\left[q_i \prod_{j \in z-i} (1 - q_i) | i \in z\right]$$

Non-cooperative

- Each mobile optimizes Θ_i^{act}
- Only NE is $q_i = 1$ ($q_i = q_i^{max}$ if power constraint)
- Throughput is π_1/m where $\pi_n = \sum_{|z|=n} \zeta(z)$
- Conditional throughput for node i $\zeta(e_i)$ and q_i^{max} if power constraint

Cooperative

- BS does not have control over mobile and no control on their power (only serve as signaling device)
- Strategy:
  - Partition nodes in K group $S_j \ni i$ for $j = 1, \ldots, K$ if $i = j - 1 mod(K)$
  - Correlated strategy: (p, q)
  - At time t, active node i transmits with proba p_i is $i \in S_{X(t)}$ where $X(t) \sim Unif(0, K)$, otherwise transmit with proba q_i
- Power constraint $Pow(p_i, q_i) = \frac{P}{K} + \frac{(K-1)q}{K} \leq q_i^{max}$
- Strategy (p_i, q_i) is a correlated equilibrium for i if it is a best response to all mobiles strategies (for Θ_i^{act} and Θ_{all})
- Goal: optimize the throughput(s) Θ_{all}
- Results: figures