A New PLL Frequency Synthesis Structure

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Abstract—In a PLL frequency synthesizer, the reference and VCO frequencies are divided down to their greatest common divisor frequency prior to phase comparison. We replace the dividers by a simplified form of digital frequency synthesizer, and examine the spectrum of the phase comparator outputs to show that the loop bandwidth can be dramatically increased, thereby improving the pull-in range and acquisition time.

1.0 INTRODUCTION

THE purpose of a frequency synthesizer is to derive from a reference frequency f_1 a new frequency $f_2 = K_2 f_1 / K_1$, where K_1 and K_2 are integers. One approach is to build a digital frequency synthesizer [1, 2], which requires that $K_2 < K_1/2$ or $f_1 > 2f_2$. The major advantage of this approach, which in some applications justifies its complexity, is that when the K_1 and K_2 are made programmable, the synthesized frequency can be changed virtually instantaneously. Another approach, which places no restrictions on K_1 and K_2 , is the phase-locked loop (PLL) frequency synthesizer of Fig. 1(a) [2–5]. The reference frequency f_1 and voltage-controlled oscillator (VCO) frequency f_2 are divided down to their greatest common divisor frequency

$$f_0 = \frac{f_1}{K_1} = \frac{f_2}{K_2} \quad (1.1)$$

for phase comparison.

Often the design of the loop filtering in Fig. 1 is a difficult task because there is a significant component at f_0 at the phase comparator output which must be highly attenuated to avoid significant f_0 sidebands on f_2 . When f_0 is much smaller than f_2 , this dictates a very narrow loop bandwidth, which results in a small pull-in range and slow acquisition [5]. One approach to increasing both pull-in range and acquisition speed is to add a frequency comparator to the phase comparator [6]. In this paper we will describe an alternative method of improving the loop acquisition properties. Specifically, we show that the loop bandwidth can be dramatically increased by modifying the divider circuits in Fig. 1.

The modified PLL synthesizer we will propose is in the same spirit as the digiphase synthesizer proposed by Gillette

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Fig. 1. PLL Frequency Synthesis. (a) PLL Synthesizer with f_0 Phase Comparison. (b) PLL Synthesizer Incorporating Digital Frequency Synthesizers.

[7]. However, our method has no restrictions on what frequency can be derived (the digiphase synthesizer requires that K_1 be of the form $2^m 5^n$ for base ten arithmetic or 2^m for base two arithmetic) and will generally require less hardware. Our work is also related to the concept of a "smooth pulse sequence" [8, 9] in that what we propose is essentially a new and simpler method of generating a smooth pulse sequence.

In Section 2.0 we describe our modified synthesizer. The bulk of the paper is then devoted to deriving the various spectra of interest in Section 3.0, with mathematical details relegated to the Appendices. This is a fairly challenging application of the principles of number theory [10]. A numerical example demonstrating the benefits of our approach is given in Section 4.0.

2.0 MODIFIED PLL SYNTHESIZER

We begin by noting that the dividers of Fig. 1(a) can be replaced by digital frequency synthesizers [1] as in Fig. 1(b). Thus, instead of simply dividing f_1 by K_1 , we can multiply it by any rational fraction L_1/N_1 , provided only that $2L_1 < N_1$. We can choose L_1 , N_1 , L_2 , and N_2 such that both inputs to the phase comparator are at frequency f_{pc} , where

$$f_{pc} = \frac{K_1 L_1}{N_1} f_0 = \frac{K_2 L_2}{N_2} f_0. \quad (2.1)$$

Now f_{pc} can be chosen to be much larger than f_0 , and since the fundamental frequency at the output of the phase comparator is f_{pc} , the loop bandwidth can be safely increased.

The only advantage of Fig. 1(b) over a digital frequency synthesizer for deriving f_2 from f_1 directly is that there is no fundamental restriction on what f_2 can be derived in this manner. Relative to the PLL synthesizer of Fig. 1(a), Fig. 1(b) has much more hardware, and requires a more complicated analog phase comparator (since the digital synthesizers have analog sinusoid outputs).

We can simplify Fig. 1(b), and return to a digital phase comparator, by replacing the digital frequency synthesizer in Fig. 1(b) by a simpler circuit to be described in the next section. It will be shown in Section 3.0 that for the simpler circuit the phase comparator output fundamental returns to f_0 , as in Fig. 1(a), but that the size of this fundamental and other low-frequency harmonics are dramatically reduced. Thus, the effect is much the same as if the loop filter had very narrow bandwidth, but without the impact on acquisition performance.

2.1 Simple Approximate Digital Frequency Synthesizer

Suppose we have available a clock at reference frequency f and we wish to derive a new frequency Lf/N . An exact digital frequency synthesizer [1, 2, 11] generates a sequence of samples

$$x_L(k) = A \sin(2\pi Lk/N) \quad (2.2)$$

which repeat with period N ,

$$x_L(k + N) = x_L(k). \quad (2.3)$$

The implication of (2.3) is that only N samples need to be stored to generate (2.2).^{*} In particular, if we store the N samples $x_1(k)$, $0 \leq k \leq N-1$, then (2.2) can be generated by the relation

$$x_L(k) = x_1(kL \bmod N). \quad (2.4)$$

In (2.4) the function $n \bmod N$ is the remainder (an integer between 0 and $N-1$) when n is divided by N .

In order to generate (2.4) by digital means we can store the samples $x_1(k)$ in a read-only memory (ROM), generate the address $M(k) = kL \bmod N$ digitally, and use a D/A converter to generate the analog sample and a low-pass filter to reconstruct the analog sinusoid. This realization is shown in Fig. 2(a). We will take a closer look at the address generator in Section 2.2. The important points to note are that, for fixed N , a) the address can be represented by n bits for the smallest n such that $N \leq 2^n$ without fear of overflow since $kL \bmod N$ is always in the range 0 to $N-1$, and b) the frequency can be changed without altering the ROM contents simply by changing L in the address generator as long as N is fixed.

The D/A conversion introduces quantization noise to the synthesized frequency. This effect is analyzed in detail in [11], where it is shown that the output consists of harmonics of f/N , by far the largest being at Lf/N , of course.

^{*} These samples could also be recursively generated [1].

Fig. 2. Digital Frequency Synthesis. (a) A Digital Frequency Synthesizer. (b) A Simplified Approximate Synthesizer.

We can eliminate the D/A converter entirely by representing $x_1(k)$ by a 1-bit quantization; that is, generate a one if $x_1(k)$ is positive or zero and generate a zero if $x_1(k)$ is negative. In fact, we can also then eliminate the ROM as well, since we see in (2.2) that

$$x_1(k) \geq 0 \text{ if } \begin{cases} 0 \leq k \leq N/2 - 1, & N \text{ even} \\ 0 \leq k \leq \frac{N-1}{2}, & N \text{ odd.} \end{cases} \quad (2.5)$$

Therefore, the ROM can be replaced, as in Fig. 2(b), by a digital comparator which compares two n -bit binary numbers** (not to be confused with a phase comparator). Although this approximation is coarse, the PLL acts to suppress the relatively large sidebands. The relevant spectra are calculated in Section 3.0. Figure 2(b) is the approximate synthesizer which we propose as a replacement for the exact synthesizer in Fig. 1(b).

At this point the reader would probably benefit from a simple example. Let $N = 8$, and suppose we desire to derive the frequency $3f/8$ where we have available a clock at rate f . Then we generate the sequence of addresses of Table 1. The resulting approximate representation of $3f/8$ is shown in Fig. 3. This waveform has three transitions for every eight transitions of f , and repeats itself after every eight transitions of f . It has a fundamental frequency $f/8$, but the largest harmonic is the third, at $3f/8$. The PLL can suppress the undesired harmonics to any extent desired.

One way to view the circuit of Fig. 2(b) is that it attempts to approximate frequency Lf/N with a waveform which has

** If N is a power of two, this comparator is not needed. The approximate sinusoid is just the most significant bit of the address.

TABLE 1

k	$M(k) = 3k \bmod 8$	$M(k) \leq 3 ?$	Digital Comparator Output
0	0	YES	1
1	3	YES	1
2	6	NO	0
3	1	YES	1
4	4	NO	0
5	7	NO	0
6	2	YES	1
7	5	NO	0
(PERIODIC REPETITION THEREAFTER)

Fig. 3. Example Waveforms for $N = 8$, $L = 3$. (a) Reference Frequency f . (b) Derived Approximate Frequency $3f/8$.

transitions corresponding to transitions of reference frequency f . In particular, it chooses the transition of f closest to the actual transition of exact frequency Lf/N . The maximum error, or jitter, is thus one half period of f , or $1/2f$ s. As f gets large relative to Lf/N , the significance of this jitter diminishes. The PLL acts to reduce this jitter.

In some applications the approximate synthesized waveform may be adequate without further jitter reduction in a PLL. For example, as a clock for a digital logic circuit, the residual jitter may be small enough to be of no consequence.

2.2 Address Generator

The address generator of Fig. 2 is readily designed from the recursive relation

$$\begin{aligned} M(k+1) &= (k+1)L \bmod N \\ &= (kL \bmod N + L) \bmod N \\ &= (M(k) + L) \bmod N. \end{aligned} \quad (2.6)$$

Writing this relationship in another way,

$$M(k+1) = \begin{cases} M(k) + L, & 0 \leq M(k) \leq N-1-L \\ M(k) + L - N, & N-L \leq M(k) \leq N-1 \end{cases}$$

we get the implementation of Fig. 4. The address $M(k)$ is stored in the clocked latch while the new address $M(k+1)$ is determined. The latter is generated by comparing $M(k)$ to $N-L$; if $M(k) < N-L$ then L is added to $M(k)$, and otherwise $L-N$ (a negative number) is added. The added complexity of this circuit relative to the divide by K_1 and K_2 counters of

Fig. 4. Address Generator.

Fig. 1(a) will of course limit the usefulness of this approach to slightly lower frequencies.

3.0 RELEVANT SPECTRA

We will now calculate the spectra of the approximate synthesized frequency of Fig. 2(b) and the output of the phase comparator of Fig. 1(b) when the approximate digital frequency synthesizer of Fig. 2(b) replaces the two exact synthesizer boxes and an exclusive-or phase detector is used. The most important result is that the fundamental of the phase comparator output is f_0 , just as in Fig. 1(a), as long as N_1 and N_2 contain no common factors other than unity (otherwise the fundamental is a subharmonic of f_0 , a situation to be avoided). A numerical example in Section 4.0 will demonstrate, however, that this fundamental and the lower frequency harmonics have been dramatically reduced relative to Fig. 1(a).

We can use the model of Fig. 5 to analyze the in-lock performance of the PLL synthesizer. The clocks f_1 and f_2 we consider to be counted down from a fictitious high-speed clock with frequency $K_1 K_2 f_0$, the least common multiple of f_1 and f_2 , in order to make the problem a simpler discrete one. This has the unimportant affect of restricting the relative phase of the two clocks to discrete multiples of $(K_1 K_2 f_0)^{-1}$ s. The inputs and outputs of the exclusive-or phase comparator*** assume the values +1 and 0, and obey the relation

$$z_k = 2r_{k+k_0}s_k - r_{k+k_0} - s_k + 1 \quad (3.1)$$

where the index k runs at frequency $K_1 K_2 f_0$ throughout. The relative phase of the two inputs is determined by k_0 .

3.1 Periodicity of Phase Detector Output

Using the notation (m, n) to denote the greatest common divisor of m and n , we henceforth assume that $(K_1, K_2) = (L_1, N_1) = (L_2, N_2) = 1$ (which we call relatively prime) since any common factors in these pairs of integers can be divided out without modifying any of the frequencies generated. We make no restriction on (L_1, L_2) and (N_1, N_2) ,

$$(L_1, L_2) = d_L \quad (3.2)$$

$$(N_1, N_2) = d_N. \quad (3.3)$$

*** A set-reset flip-flop could also be used with similar results.

Fig. 5. Model of an In-Lock PLL.

It follows that $(d_L, d_N) = 1$. From (2.1) there is the constraint

$$1 = \frac{K_1 L_1 N_2}{K_2 N_1 L_2} = \frac{K_1}{K_2} \left[\frac{L_1}{d_L} \right] \left[\frac{N_2}{d_N} \right] \left[\frac{N_1}{d_N} \right] \left[\frac{L_2}{d_L} \right]$$

which through factor analysis can be shown to imply that

$$K_1 = \frac{N_1 L_2}{d_N d_L} \quad (3.4)$$

$$K_2 = \frac{L_1 N_2}{d_N d_L} \quad (3.5)$$

It is evident that r_k is periodic in $K_2 N_1$, while s_k is periodic in $K_1 N_2$, and hence the minimum period N of z_k must satisfy

$$N = \alpha K_2 N_1 = \beta K_1 N_2$$

for relatively prime integers α and β . Hence, from (3.4) and (3.5),

$$\frac{\alpha}{\beta} = \frac{K_1 N_2}{K_2 N_1} = \frac{L_2}{L_1}$$

and it follows that

$$\alpha = L_2 / d_L$$

$$\beta = L_1 / d_L$$

$$N = K_2 L_2 N_1 / d_L = K_1 L_1 N_2 / d_L. \quad (3.6)$$

The fundamental frequency of z_k is therefore

$$\frac{K_1 K_2 f_0}{N} = \frac{f_0}{d_N} \quad (3.7)$$

so that there is a strong motivation to keep $d_N = 1$ (N_1 and N_2 relatively prime).

3.2 Spectra of Phase Detector Inputs

The discrete Fourier transform (DFT) of the input sequence r_k is

$$R_p = \frac{1}{K_2 N_1} \sum_{k=0}^{K_2 N_1 - 1} r_k e^{-j2\pi k p / K_2 N_1}. \quad (3.8)$$

A similar relationship holds for S_p , the DFT of s_k , with $K_1 N_2$ substituted for $K_2 N_1$. It is shown in Appendix A that for $p \neq 0$,

$$R_p = \frac{[1 - e^{-j2\pi p / N_1}] [1 - (-1)^{p J_1} z]}{K_2 N_1 [1 - e^{-j2\pi p / K_2 N_1}] [1 - e^{-j2\pi p J_1 / N_1}]} \quad (3.9)$$

where

$$z = \begin{cases} 1, & N_1 \text{ even} \\ e^{-j\pi p J_1 / N_1}, & N_1 \text{ odd} \end{cases} \quad (3.10)$$

and where J_1 is the unique integer in $[0, N_1 - 1]$ which satisfies

$$J_1 L_1 \bmod N_1 = 1. \quad (3.11)$$

In addition, when $p = 0$,

$$R_0 = \begin{cases} 1/2, & N_1 \text{ even} \\ (N_1 + 1)/2N_1, & N_1 \text{ odd} \end{cases} \quad (3.12)$$

3.3 Spectrum of Phase Detector Output

It is shown in Appendix B that the DFT of z_k is

$$\begin{aligned} Z_p &= \frac{1}{N} \sum_{k=0}^{N-1} z_k e^{-j2\pi k p / N} \\ &= \delta_{p,0} + \sum_{k=0}^{K-1} e^{j2\pi k_0 p_1 / K_2 N_1} R_{p_1} S_{p_2} \\ &\quad - e^{j2\pi k_0 p / N} R_{d_L p / L_2} M_2(p) - S_{d_L p / L_1} M_1(p) \end{aligned} \quad (3.13)$$

where

$$K = K_2 N_1 d_L / L_1 \quad (3.14)$$

$$p_1 = p_1^* + L_1 k / d_L \quad (3.15)$$

$$p_2 = (p_2^* - L_2 k / d_L) \bmod K_1 N_2 \quad (3.16)$$

and $0 \leq p_1^* < L_1 / d_L$ and p_2^* uniquely satisfies

$$L_2 p_1^* / d_L = p \bmod L_1 / d_L \quad (3.17)$$

and

$$p_2^* = \frac{d_L p - L_2 p_1^*}{L_1} \quad (3.18)$$

and the function $M_k(p)$ is unity if $d_L p/L_k$ is an integer and zero otherwise.

4.0 NUMERICAL EXAMPLE

So far we have not discussed how the integers L_1, N_1, L_2, N_2 should be chosen other than that they must satisfy (2.1) and that it is advantageous if $d_N = 1$. Ideally they would be chosen to minimize the jitter on derived frequency f_2 for some desired loop filter. A simpler strategy, which works as long as one of f_1 or f_2 is more than twice as large as the other, is to choose $L_1 = N_1 = 1$ if $f_2 > 2f_1$ or $L_2 = N_2 = 1$ if $f_1 > 2f_2$. This has the advantage of eliminating one of the approximate digital frequency synthesizers.

As a numerical example, we consider the case $K_1 = 193$ and $K_2 = 2048$ (this example is motivated by two frequencies prevalent in the Bell System digital transmission and switching networks). It follows that $L_2 = K_1$ and $N_2 = K_2$ if we choose $L_1 = N_1 = 1$: Note that in this case the reference f_1 is the lower frequency, the fundamental frequency at the phase detector output is $f_0 = f_1/193$, and the nominal frequency at the phase detector input is $f_{pc} = f_1 = 193f_2/2048$.

At the output of the phase detector, the fundamental f_0 and lower harmonics were found to have little dependence on the phase k_o , so the higher harmonics were examined for $k_o = 512$ using (3.13). The first very significant harmonic is $386f_0$ (the second harmonic of f_1), for which $C_{386} = -10.5$ dB, where

$$C_k = 20 \log_{10} |Z_k|.$$

For most $k < 386$, C_k is in the range of -40 dB or greater, with odd harmonics including f_0 virtually nonexistent. The only exceptions are $C_{118} = -30$ dB, $C_{150} = -37$ dB, $C_{236} = -36$ dB, and $C_{268} = -31$ dB. This performance is quite favorable when compared with that of Fig. 3(a), where $C_1 = -10$ dB. The addition of the approximate synthesizer therefore adds more than 30 dB additional suppression of all harmonics out to the 118th thereby greatly easing the loop filtering requirement.

5.0 CONCLUSION

The modification of the PLL synthesizer we have described would appear to deserve consideration in many applications. The increase in circuit complexity when two dividers are replaced by a single address generator and comparator is not great, and programmability of the synthesized frequency is just as simple. The benefits in terms of improved acquisition and/or reduced sidebands on the synthesized frequency have been demonstrated.

APPENDIX A

It is convenient to write the two-level sinusoid approximation as

$$x_1(k) = \begin{cases} \psi_{[0, N/2-1]}^{(k)}, & N \text{ even} \\ \psi_{[0, (N-1)/2]}^{(k)}, & N \text{ odd} \end{cases} \quad (\text{A.1})$$

where the indicator function ψ is defined as

$$\psi_s(k) = \begin{cases} 1, & k \in S \\ 0, & k \notin S \end{cases} \quad (\text{A.2})$$

Equations for r_k and s_k can then be obtained as follows: Divisions of the $K_1 K_2 f_0$ clock by K_1 and K_2 are obtained by defining new indexes m and n

$$\begin{aligned} k &= mK_2 + i, & 0 \leq i \leq K_2 - 1 \\ k &= nK_1 + j, & 0 \leq j \leq K_1 - 1 \end{aligned} \quad (\text{A.3})$$

from which, from (A.1) and (2.4),

$$\begin{aligned} r_k &= \psi_{[0, N_1/2-1]}(mL_1 \bmod N_1) \\ s_k &= \psi_{[0, N_2/2-1]}(nL_2 \bmod N_2) \end{aligned} \quad (\text{A.4})$$

for N even, with similar equations for N odd.

Expression (3.9) will now be derived. Substituting (A.3) and (A.4) into (3.8)

$$\begin{aligned} R_p &= \frac{1}{K_2 N_1} \sum_{i=0}^{K_2-1} \sum_{m=0}^{N_1-1} \psi_{[0, N_1/2-1]}(mL_1 \bmod N_1) \\ &\quad \cdot e^{-j2\pi(mK_2+i)p/K_2 N_1} \\ &= \frac{1}{K_2 N_1} \frac{1 - e^{-j2\pi p/N_1}}{1 - e^{-j2\pi p/K_2 N_1}} \sum_{m=0}^{N_1-1} \psi_{[0, N_1/2-1]} \\ &\quad \cdot (mL_1 \bmod N_1) e^{-j2\pi m p/N_1}. \end{aligned} \quad (\text{A.5})$$

The remaining sum can be evaluated by letting

$$q = mL_1 \bmod N_1 \quad (\text{A.6})$$

which has inverse relationship [11]

$$m = qJ_1 \bmod N_1 \quad (\text{A.7})$$

where J_1 is given by (3.11). Then the sum in (A.5) becomes

$$\sum_{q=0}^{N_1-1} \psi_{[0, N_1/2-1]}(q) e^{-j2\pi q p J_1/N_1}$$

and (3.9) follows immediately. A minor modification yields a similar formula for N odd.

APPENDIX B

The DFT of (3.13) will now be evaluated for the z_k of (3.1). Substituting (3.8) and a similar equation for S_p into (3.1) and (3.13)

$$Z_p = \delta_{p,0} + 2 \sum_{p_1=0}^{K_2 N_1 - 1} \sum_{p_2=0}^{K_1 N_2 - 1} e^{j2\pi k_0 p_1 / K_2 N_1} \cdot R_{p_1} S_{p_2} \psi_{I_1}(p_1, p_2) - e^{j2\pi k_0 p / N} R_{d_L p / L_2} \psi_{I_2}(p) - S_{d_L p / L_1} \psi_{I_3}(p) \quad (\text{B.1})$$

where

$$I_1 = \left\{ p_1, p_2 \left| \frac{L_2 p_1 + L_1 p_2}{d_L} = p \bmod N \right. \right\} \quad (\text{B.2})$$

$$I_2 = \{p | p = 0 \bmod L_2 / d_L\} \quad (\text{B.3})$$

$$I_3 = \{p | p = 0 \bmod L_1 / d_L\}. \quad (\text{B.4})$$

The first sum in (B.1) can be simplified considerably by a better characterization of the index set I_1 . Specifically, it is shown in Appendix C that I_1 of (B.2) is given equivalently by (3.15) and (3.16) where $0 \leq k \leq K_2 N_1 d_L / L_1 - 1$. Thus, the double summation over p_1 and p_2 becomes a single summation over k . The result is a reduction by a factor of $L_1 L_2 / d_L^2$ in the computation required to evaluate the summation (this factor is 193 for the numerical example of Section 4.0).

APPENDIX C

If we rewrite the specification of I_1 in (B.2) as

$$\frac{L_2 p_1 + L_1 p_2}{d_L} = p + nN \quad (\text{C.1})$$

for some n and fix n , we recognize (C.1) as a first degree Diophantine equation [10, p. 96]. All solutions for p_1 are of the form of (3.15) where p_1^* is any particular solution. To verify that the unique p_1^* satisfying (3.17) is a solution to (C.1) for any n , note that since L_1 / d_L divides N

$$\begin{aligned} L_2 p_1^* / d_L &= p \bmod L_1 / d_L \\ &= (p + nN) \bmod L_1 / d_L \\ &= p + nN + dL_1 / d_L \end{aligned}$$

which is of the form of (C.1) with $p_2 = -d$. The fact that p_1^* is unique follows from the fact that L_1 / d_L and L_2 / d_L are relatively prime. To establish the range of k in (3.15), observe that since

$$0 \leq p_1^* \leq L_1 / d_L - 1$$

and since in (B.1) the range of p_1 becomes, substituting from (3.15),

$$0 \leq p_1^* + L_1 k / d_L \leq K_2 N_1 - 1$$

or

$$-1 < -\frac{p_1^* d_L}{L_1} \leq k \leq \frac{K_2 N_1 d_L}{L_1} - \frac{[p_1^* + 1] d_L}{L_1}.$$

Since

$$0 < \frac{(p_1^* + 1) d_L}{L_1} \leq 1$$

it follows that

$$0 \leq k \leq \frac{K_2 N_1 d_L}{L_1} - 1.$$

Finally, when (3.15) is substituted into (C.1) we can solve for p_2 ,

$$p_2 = \frac{d_L p - L_2 p_1^*}{L_1} - \frac{L_2}{d_L} k + nK_1 N_2 \quad (\text{C.3})$$

which in view of the range of p_2 in (B.1) is satisfied for precisely one n , that given by (3.16) and (3.18).

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David G. Messerschmitt (S'65-M'68), for a photograph and biography, see page 675 of the May 1978 issue of this TRANSACTIONS.