EECS 225A Spring 2005

Homework 9 solutions

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1. Given an $N \times N$ autocorrelation matrix $\mathbf{R}^{(\mathbf{N})}$, you are told that the eigenvalues of this matrix, asymptotically as $N \to \infty$, approach the values ($|a|<1$ is real-valued)

$$\frac{1}{1+a^2-2a\cdot\cos\left(\frac{2\pi n}{N}\right)}, \quad 0 \le n \le N-1$$

a. Identify a general family of autocorrelation functions $r(m)$ whose autocorrelation matrix has these eigenvalues, asymptotically. (By 'family', we mean more than one autocorrelation function, perhaps indexed by some parameter.) You can assume that the power spectrum $P(e^{j\omega})$ is a "well behaved" function (continuous, bounded variation, etc.). You may have to make one or more "reasonable" assumptions; state any such assumptions.

b. Find a *provable* upper bound on the eigenvalue spread of $\mathbf{R}^{(\mathbf{N})}$ that is valid for all N (subject to any assumptions you had to make in part a.). (HINT: You don't need to actually prove anything; the proof is already in Hayes.)

c. Working with this upper bound on the eigenvalue spread of b., for the step size in the steepest descent algorithm that maximizes the rate of convergence of the slowest-converging mode, what is the time constant of the convergence of the slowest-converging mode? The time constant is defined as the number of time increments that it takes for this mode of the filter coefficient offset to decrease to e^{-1} of its initial value.

c. Evaluate and plot the actual (exact) eigenvalue spread for $\mathbf{R}^{(\mathbf{N})}$ numerically for one representative member of the family of autocorrelation functions found in a. using Matlab for $2 \le N \le 200$ and compare the actual eigenvalue spread to the bound you developed in b.

Solution

a. From results derived in class, we know that the sampled power spectrum $P(W_N^n)$, $0 \le n \le N$-1 assumes precisely the asymptotic eigenvalues. What we don't know is the *order* in which the power spectrum assumes these values since the eigenvalues (unlike the power spectrum) have no natural ordering. Since the values given, in the order given, are the uniformly spaced samples of the well-behaved function

$$\frac{1}{1+a^2-2a\cdot\cos(\omega)}, \quad 0\le\omega<2\pi$$

it is natural to assume that this is the power spectrum. However, we must ask: Is there any other obvious ordering of the eigenvalues (which, after all, have no natural ordering) that converge to a well-behaved valid power spectrum? Remember that the only requirement on the power spectrum is that it be a non-negative function of frequency, and in addition we are asking that it be a well-behaved continuous function. One obvious answer is the family of power spectra (indexed by parameter θ)

$$P_\theta(e^{j\omega})=\frac{1}{1+a^2-2a\cdot\cos(\omega+\theta)}, \quad 0\le\omega<2\pi\ .$$

All members of this family are continuous at all frequencies (including at $\omega=0$) and have the same set of asymptotic eigenvalues for $\mathbf{R}^{(\mathbf{N})}$; namely, the given set. For most members of this family the power spectrum is asymmetric about $\omega=0$, so the resulting process is not real-valued. Rewriting this power spectrum as (where $b_\theta=a\cdot e^{-j\theta}$),

$$P_\theta(e^{j\omega})=\frac{1}{(1-b_\theta\cdot e^{-j\omega})(1-b_\theta^*\cdot e^{j\omega})}=\frac{1}{\left|1-b_\theta\cdot e^{-j\omega}\right|^2}$$

$$P_\theta(e^{j\omega})=\frac{1}{(1-b_\theta\cdot z^{-1})(1-b_\theta^*\cdot z)}$$

we get a family of autocorrelation functions

$$r_\theta(m)=\frac{a^{|m|}\cdot e^{-j\theta m}}{1-a^2}\ .$$

This autocorrelation function is real-valued when (and only when) $\theta=\{0,\pi\}$.

b. Hayes p.97 shows that the power spectrum extremes bound the eigenvalues of $\mathbf{R}^{(\mathbf{N})}$; thus

$$\frac{1}{(1+a)^2}\le\lambda\le\frac{1}{(1-a)^2}\ \text{for the range}\ 0<a<1.$$

Thus, for this range the eigenvalue spread is similarly bounded by

$$s=\frac{\lambda_{\max}}{\lambda_{\min}}\le\frac{(1+a)^2}{(1-a)^2}\quad\text{or}\quad\frac{\sqrt{s}-1}{\sqrt{s}+1}\le a\ .$$

Note that these values are independent of θ, which makes sense since θ should not affect the distribution of eigenvalues, only (possibly) their ordering. Also note that as $a \to 1$, the slowest mode of convergence progressively slows down.

c. The convergence of the slowest mode is, as shown in lecture, the time constant τ is the smallest τ for which

$$a^{\tau} \le e^{-1} \quad \text{or} \quad \tau \ge \frac{-1}{\log a}.$$

(Why does the inequality reverse? Because $\log(a) < 0$.)

d. We choose $a = 0.9$ and (to confirm the premise that θ makes no difference) $\theta = \pi/3$. Here is the actual eigenvalue spread from $N = 2$ to $N = 200$. By comparison, the power spectrum bound on eigenvalue spread is $s \le 361$. Note that the eigenvalue spread increases monotonically with N as predicted by the Bordering Theorem (Hayes p. 48). This implies that the steepest descent algorithm convergence deteriorates as N increases. The matrix has to reach size $N \approx 100$ before the actual eigenvalue spread approximates the power spectrum bound.