EECS 225A Spring 2005

Homework 7 solutions

1. You wish to design a least-squares inverse filter that realizes (or if necessary approximates) $g(k) \otimes h_N(k) = d(k)$, $0 \le k < M$. However, battery power limitations restrict the value of N (number of FIR inverse filter coefficients) to $N = 2$. You can assume that $\sum_{k=0}^{\infty} |g(k)|^2 < \infty$. In the context of your application, the accuracy of overall unit-sample response of the cascade of filter plus inverse filter only matters for $0 \le k < M$.
   a. Formulate the equations you would need to solve for $M = 2$.
   b. Repeat a. for $M = 3$.
   c. Use Matlab to numerically calculate the inverse filter and resulting unit-sample response of the filter cascade for the a. and b. cases. Assume that $g(k)$ is real-valued and that

$$g(k) = \begin{cases} 1/(1+k)^2, & 0 \le k \le 10 \\ 0, & k > 10 \end{cases}.$$

Solution

a. Since the numerical example in c. is real-valued, let us assume the LS inverse and desired signals are real-valued. We then have to solve the linear equations $\mathbf{Ah} = \mathbf{d}$ where

$$\mathbf{A} = \begin{bmatrix} g(0) & 0 \\ g(1) & g(0) \end{bmatrix}, \ \mathbf{h} = \begin{bmatrix} h_2(0) \\ h_2(1) \end{bmatrix}, \text{ and } \mathbf{d} = \begin{bmatrix} d(0) \\ d(1) \end{bmatrix}.$$

$\mathbf{A}$ is non-singular iff $g(0) \ne 0$.

b. In this case,

$$\mathbf{A} = \begin{bmatrix} g(0) & 0 \\ g(1) & g(0) \\ g(2) & g(1) \end{bmatrix}, \ \mathbf{h} = \begin{bmatrix} h_2(0) \\ h_2(1) \end{bmatrix}, \text{ and } \mathbf{d} = \begin{bmatrix} d(0) \\ d(1) \\ d(2) \end{bmatrix}.$$

$\mathbf{A}$ is fortunately always full rank. The system is over-determined, so we rely on the pseudo-inverse. The equations to be solved become $\mathbf{A^T A h} = \mathbf{A^T d}$ where

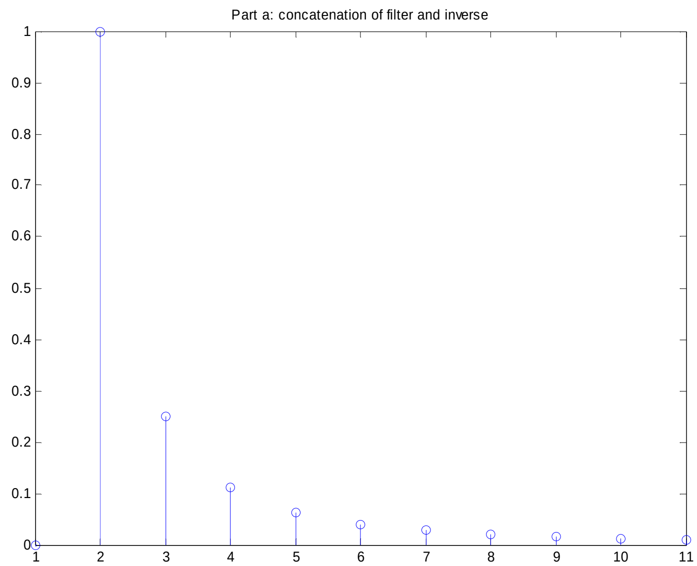
$$\mathbf{A^T A} = \begin{bmatrix} g^2(0) + g^2(1) + g^2(2) & g(0)g(1) + g(1)g(2) \\ g(0)g(1) + g(1)g(2) & g^2(0) + g^2(1) \end{bmatrix} \text{ and}$$

$$\mathbf{A^T d} = \begin{bmatrix} g(0)d(0) + g(1)d(1) + g(2)d(2) \\ g(0)d(1) + g(1)d(2) \end{bmatrix}$$

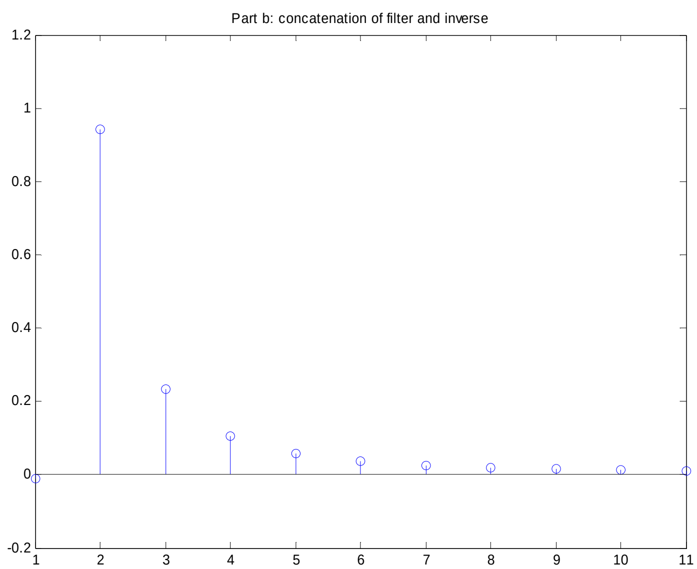
c. Sorry I forgot to specify $d(k)$, but let's make an assumption that the desired signal is

$$\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ (part a.) } \mathbf{d} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ (part b.)}$$

This gives the filter design a delay to work with, and using the same delay in both cases allows us to see the impact of increasing the modeling interval. See hmwk07.m for the program. The concatenated filters have the following impulse response in parts a. and b.:



Note that the modeling is exact, but of course the filter is far from being a simple delay because we did not care about the later unit-sample response values.



Now the modeling is not exact because we do not have sufficient degrees of freedom. The solution has decreased the second value in order also bring the third closer to zero.

2. Hayes problem 5.2

Solution

(a) If $|\Gamma_j| < 1$ for $j = 1, 2, \ldots, p-1$, then the zeros of $A_{p-1}(z)$ are *inside* the unit circle. With $|\Gamma_p| = 1$, let $\Gamma_p = e^{j\theta}$. We then have

$$A_p(z) = A_{p-1}(z) + e^{j\theta} z^{-p} A_{p-1}^*(1/z^*)$$

Conjugating and replacing z with $1/z^*$,

$$A_p^*(1/z^*) = A_{p-1}^*(1/z^*) + e^{-j\theta} z^{p} A_{p-1}(z)$$

Multiplying both sides of this equation by $e^{j\theta} z^{-p}$ gives

$$e^{j\theta} z^{-p} A_p^*(1/z^*) = A_{p-1}(z) + e^{j\theta} z^{-p} A_{p-1}^*(1/z^*)$$

which is equal to $A_p(z)$. Thus, with

$$A_p(z) = e^{j\theta} z^{-p} A_p^*(1/z^*)$$

it follows that if $A_p(z)$ has a zero at $z = z_0$, then $A_p(z)$ must also have a zero at $z = 1/z_0^*$. Now, recall that if $|\Gamma_j| < 1$ for $j = 1, 2, \ldots, p$ then the zeros of $A_p(z)$ must lie *inside* the unit circle, no matter how close $|\Gamma_p|$ may be to one. Since the zeros of $A_p(z)$ move continuously as Γ_p is varied, the zeros of $A_p(z)$, which lie in reciprocal pairs when $\Gamma_p = 1$, must all be on the unit circle.

(b) By definition, we have

$$\begin{aligned} A_p(z) &= A_{p-1}(z) + \Gamma_p z^{-p} A_{p-1}^*(1/z^*) \\ \widetilde{A}_p(z) &= A_{p-1}(z) + (1/\Gamma_p^*) z^{-p} A_{p-1}^*(1/z^*) \end{aligned}$$

Multiplying both sides of the expression for $\widetilde{A}_p(z)$ by Γ_p^* yields

$$\Gamma_p^* \widetilde{A}_p(z) = \Gamma_p^* A_{p-1}(z) + z^{-p} A_{p-1}^*(1/z^*)$$

Conjugating and replacing z with $1/z^*$ this becomes

$$\Gamma_p \widetilde{A}_p^*(1/z^*) = \Gamma_p A_{p-1}^*(1/z^*) + z^{p} A_{p-1}(z)$$

Multiplying both sides by z^{-p} gives

$$z^{-p} \Gamma_p \widetilde{A}_p^*(1/z^*) = z^{-p} \Gamma_p A_{p-1}^*(1/z^*) + A_{p-1}(z)$$

and we see that the right-hand side is equal to $A_p(z)$. Therefore,

$$A_p(z) = \Gamma_p z^{-p} \widetilde{A}_p^*(1/z^*)$$

and it follows that $\widetilde{A}_p^*(1/z^*)$ is equal to zero when $A_p(z)$ is equal to zero. In other words, the zeros of $A_p(z)$ are *reflected* about the unit circle, so that a zero at $z = z_0$ in $A_p(z)$ becomes a zero at $z = 1/z_0^*$ in $\widetilde{A}_p(z)$.

(c) As Γ_p increases from $\Gamma_p = \epsilon$ to $\Gamma_p = 1$, the zeros of $A_p(z)$ move *towards* the unit circle. When $\Gamma_p = 1$ all of the zeros lie on the unit circle. As Γ_p increases beyond 1, the zeros move outside the unit circle and approach their mirror image location as $\Gamma_p \to 1/\epsilon$.

3. Hayes problem 5.3

Solution

(a) We want to show that, if $\widehat{\Gamma}_k = (-1)^k \Gamma_k$, then

$$\widehat{a}_p(k) = (-1)^k a_p(k)$$

or, if we let $\widehat{A}_p(z)$ and $A_p(z)$ be the z-transforms of $\widehat{a}_p(k)$ and $a_p(k)$, respectively, we want to show that

$$\widehat{A}_p(z) = A_p(-z)$$

We begin by noting that, for $p = 1$, we have

$$A_1(z) = 1 + \Gamma_1 z^{-1}$$

and

$$\widehat{A}_1(z) = 1 - \Gamma_1 z^{-1} = A_1(-z)$$

Therefore, let us assume that $\widehat{A}_{p-1}(z) = A_{p-1}(-z)$, and show that $\widehat{A}_p(z) = A_p(-z)$. From the Levinson order update equation we have

$$A_p(z) = A_{p-1}(z) + \Gamma_p z^{-p} A^*_{p-1}(1/z^*)$$

and

$$\begin{aligned}
\widehat{A}_p(z) &= \widehat{A}_{p-1}(z) + \widehat{\Gamma}_p z^{-p} \widehat{A}^*_{p-1}(1/z^*) \\
&= \widehat{A}_{p-1}(z) + (-1)^p \Gamma_p z^{-p} \widehat{A}^*_{p-1}(1/z^*)
\end{aligned}$$

Thus,

$$\widehat{A}_p(z) = A_{p-1}(-z) + \Gamma_p (-z)^{-p} A^*_{p-1}(-1/z^*) = A_p(-z)$$

and we have the desired result.

(b) If $\widehat{\Gamma}_k = \alpha^k \Gamma_k$ with $|\alpha| = 1$, then we may write

$$\widehat{\Gamma}_k = e^{jk\theta}\Gamma_k$$

for some real number θ. As in part (a), for $p = 1$ we have

$$A_1(z) = 1 + \Gamma_1 z^{-1}$$

and

$$\widehat{A}_1(z) = 1 + e^{j\theta}\Gamma_1 z^{-1} = A_1(e^{-j\theta}z)$$

Therefore, let us assume that $\widehat{A}_{p-1}(z) = A_{p-1}(e^{-j\theta}z)$. From the Levinson order update equation we have

$$A_p(z) = A_{p-1}(z) + \Gamma_p z^{-p} A^*_{p-1}(1/z^*)$$

and

$$\begin{aligned}\widehat{A}_p(z) &= \widehat{A}_{p-1}(z) + \widehat{\Gamma}_p z^{-p}\widehat{A}^*_{p-1}(1/z^*)\\ &= \widehat{A}_{p-1}(z) + e^{jp\theta}\Gamma_p z^{-p}\widehat{A}^*_{p-1}(1/z^*)\end{aligned}$$

Thus,

$$\widehat{A}_p(z) = A_{p-1}(e^{-j\theta}z) + e^{jp\theta}\Gamma_p z^{-p} A^*_{p-1}(e^{-j\theta}/z^*) = A_p(e^{-j\theta}z)$$

and we have the desired result,

$$\widehat{A}_p(z) = A_p(e^{-j\theta}z)$$

If $|\alpha| < 1$, then the coefficients change in no predictable manner. Consider, for example, the case of a second-order model,

$$\mathbf{a}_2 = \begin{bmatrix} 1 \\ \Gamma_1(1+\Gamma_2) \\ \Gamma_2 \end{bmatrix}$$

If $\widehat{\Gamma}_k = \alpha^k\Gamma_k$ and α is real, then

$$\widehat{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ \alpha\Gamma_1(1+\alpha^2\Gamma_2) \\ \alpha^2\Gamma_2 \end{bmatrix}$$

4. Hayes problem 5.5

Solution

Solution
The reflection coefficients corresponding to the numerator coefficients, $\mathbf{b} = [1, 0.8, -0.9, 0.3]^T$ is

$$\mathbf{\Gamma} = [-4.6522,\ -1.2527,\ 0.3]^T$$

Since $|\Gamma_1| > 1$ and $|\Gamma_2 > 1$, then the numerator polynomial is not minimum phase (there is at least one root outside the unit circle). However, since the reflection coefficients of the denominator polynomial are $\mathbf{\Gamma} = [-0.4545,\ 0.4667,\ -0.5]^T$. then the filter is stable.