EECS 225A Spring 2005

Homework 6 solutions

1. Hayes problem 4.7

Solution

(a) Note that $E(z) = Y(z)A(z) - X(z)B(z)$, so

$$e(n) = \sum_{k=0}^{p} a(k)y(n-k) - \sum_{k=0}^{q} b(k)x(n-k)$$

With

$$\mathcal{E} = \sum_{n=0}^{\infty} e^2(n)$$

the *Normal Equations* are found by setting the derivatives of $\mathcal{E}$ with respect to $a(k)$ and $b(k)$ equal to zero,

$$\frac{\partial \mathcal{E}}{\partial a(k)} = 0 \quad ; \quad \frac{\partial \mathcal{E}}{\partial b(k)} = 0$$

Thus,

$$\frac{\partial \mathcal{E}}{\partial a(k)} = \sum_{n=0}^{\infty} 2e(n)y(n-k) = 2\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{p} a(l)y(n-l) - \sum_{l=0}^{q} b(l)x(n-l)\right\} y(n-k) = 0$$

and

$$\frac{\partial \mathcal{E}}{\partial b(k)} = -\sum_{n=0}^{\infty} 2e(n)x(n-k) = -2\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{p} a(l)y(n-l) - \sum_{l=0}^{q} b(l)x(n-l)\right\} x(n-k) = 0$$

Dividing by two, and rearranging the sums, we have

$$\sum_{l=0}^{p} a(l)\left\{\sum_{n=0}^{\infty} y(n-l)y(n-k)\right\} - \sum_{l=0}^{q} b(l)\left\{\sum_{n=0}^{\infty} x(n-l)y(n-k)\right\} = 0 \quad ; \quad k = 1,\ldots,p$$

and

$$-\sum_{l=0}^{p} a(l)\left\{\sum_{n=0}^{\infty} y(n-l)x(n-k)\right\} + \sum_{l=0}^{q} b(l)\left\{\sum_{n=0}^{\infty} x(n-l)x(n-k)\right\} = 0 \quad ; \quad k = 0,\ldots,q$$

If we define

$$\begin{aligned} r_{xy}(k,l) &= \sum_{n=0}^{\infty} x(n-l)y(n-k) \\ r_y(k,l) &= \sum_{n=0}^{\infty} y(n-l)y(n-k) \\ r_x(k,l) &= \sum_{n=0}^{\infty} x(n-l)x(n-k) \end{aligned}$$

then these equations become

$$\begin{aligned} \sum_{l=0}^{p} a(l)r_y(k,l) - \sum_{l=0}^{q} b(l)r_{xy}(k,l) &= 0 \quad ; \quad k = 1,2,\ldots,p \\ -\sum_{l=0}^{p} a(l)r_{yx}(k,l) + \sum_{l=0}^{q} b(l)r_x(k,l) &= 0 \quad ; \quad k = 0,1,\ldots,q \end{aligned}$$

Assuming that the coefficients have been normalized so that $a(0) = 1$, we have

$$\begin{aligned} \sum_{l=1}^{p} a(l)r_y(k,l) - \sum_{l=0}^{q} b(l)r_{xy}(k,l) &= -r_y(k,0) \quad ; \quad k = 1,2,\ldots,p \\ -\sum_{l=1}^{p} a(l)r_{yx}(k,l) + \sum_{l=0}^{q} b(l)r_x(k,l) &= r_{yx}(k,0) \quad ; \quad k = 0,1,\ldots,q \end{aligned}$$

Writing these in matrix form we obtain

$$\begin{bmatrix} \mathbf{R}_y & -\mathbf{R}_{xy} \\ -\mathbf{R}_{yx} & \mathbf{R}_x \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_x \\ \mathbf{r}_{xy} \end{bmatrix}$$

where $\mathbf{a}^T = [a(1), a(2), \ldots, a(p)]$, $\mathbf{b}^T = [b(0), b(1), \ldots, b(q)]$, $\mathbf{r}_x^T = [r_x(1,0), r_x(2,0), \ldots, r_x(p,0)]$, and $\mathbf{r}_{xy}^T = [r_{xy}(1,0), r_{xy}(2,0), \ldots, r_{xy}(q,0)]$. Also, $\mathbf{R}_x$ is a $p \times p$ matrix with entries $r_x(k,l)$, $\mathbf{R}_y$ is a $(q+1) \times (q+1)$ matrix with entries $r_q(k,l)$, and $\mathbf{R}_{xy}$ is a $p \times (q+1)$ matrix with entries $r_x(k,l)$,

(b) Suppose $S(z) = C(z)/D(z)$. Then

$$E(z) = B(z)X(z) - \frac{C(z)}{D(z)}A(z)X(z)$$

and the error can be made equal to zero if

$$B(z) = \frac{C(z)}{D(z)}A(z) \quad \text{or} \quad \frac{B(z)}{A(z)} = \frac{C(z)}{D(z)}$$

2. Hayes problem 4.10

Solution

The equations for the coefficients $a_p(k)$, $k = 1, \ldots p$, that minimize the error $\mathcal{E}_p$ are found by setting the derivatives of $\mathcal{E}_p$ with respect to $a_p(k)$ equal to zero. Thus, assuming that $x(n)$ is real, we have

$$\frac{\partial \mathcal{E}_p}{\partial a_p(k)} = \sum_{n=0}^{\infty} 2e(n)x(n-k-N) = 0$$

Dividing by two, and substituting for $e(n)$, we have

$$\sum_{n=0}^{\infty}\left[x(n) + \sum_{l=1}^{p} a_p(l)x(n-l-N)\right] x(n-k-N) = 0$$

or

$$\sum_{l=1}^{p} a_p(l)\left[\sum_{n=0}^{\infty} x(n-l-N)x(n-k-N)\right] = -\sum_{n=0}^{\infty} x(n)x(n-k-N)$$

If we define

$$r_x(k,l) = \sum_{n=0}^{\infty} x(n-l)x(n-k)$$

then it is easily shown that $r_x(k,l)$ depends only on the difference, $k-l$, and we may write

$$r_x(k) = \sum_{n=0}^{\infty} x(n)x(n-k)$$

Thus, the normal equations become

$$\sum_{l=1}^{p} a_p(l)r_x(k-l) = -r_x(k+N)$$

Finally, using the orthogonality condition

$$\sum_{n=0}^{\infty} e(n)x(n-k-N) = 0$$

we have, for the minimum error,

$$\{\mathcal{E}_p\}_{\min} = \sum_{n=0}^{\infty} e(n)\left[x(n) + \sum_{l=1}^{p} a_p(l)x(n-l-N)\right] = \sum_{n=0}^{\infty} e(n)x(n)$$

Therefore,

$$\begin{aligned}\{\mathcal{E}_p\}_{\min} &= \sum_{n=0}^{\infty}\left[x(n) + \sum_{l=1}^{p} a_p(l)x(n-l-N)\right] x(n) \\ &= r_x(0) + \sum_{l=1}^{p} a_p(l)r_x(l+N)\end{aligned}$$

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3. Hayes problem 4.21

Solution

Solution

If we define $a_p(0) = 1$, then the error $e(n)$ is

$$e(n) = a_p(n) * x(n) - b(n) * p_{n_0}(n) = \sum_{l=0}^{p} a_p(l)x(n-l) - b(0)\left[\delta(n) + \delta(n-n_0)\right]$$

and the mean-square error that we want to minimize is

$$\mathcal{E}_p = \sum_{n=0}^{2n_0-1} e^2(n) = \sum_{n=0}^{2n_0-1} \left[\sum_{l=0}^{p} a_p(l)x(n-l) - b(0)\delta(n) - b(0)\delta(n-n_0)\right]^2$$

Setting the derivative with respect to $a_p(k)$ equal to zero, we have

$$\frac{\partial \mathcal{E}_p}{\partial a_p(k)} = \sum_{n=0}^{2n_0-1} 2\left[\sum_{l=0}^{p} a_p(l)x(n-l) - b(0)\delta(n) - b(0)\delta(n-n_0)\right] x(n-k) = 0$$

If we define

$$r_x(k,l) = \sum_{n=0}^{2n_0-1} x(n-l)x(n-k)$$

then the normal equations become (recall that $a_p(0) = 1$)

$$\sum_{l=1}^{p} a_p(l) r_x(k,l) - b(0)x(-k) - b(0)x(n_0-k) = -r_x(k,0) \quad ; \quad k = 1, 2, \ldots, p$$

Assuming that $x(n) = 0$ for $n < 0$, with $\mathbf{x} = [x(n_0-1), x(n_0-2), \ldots, x(n_0-p)]^T$, the normal equations may be written in matrix form as follows

$$\mathbf{R}_x \mathbf{a} - b(0)\mathbf{x} = -\mathbf{r}_x$$

Finally, differentiating with respect to $b(0)$ we have

$$\frac{\partial \mathcal{E}}{\partial b(0)} = -\sum_{n=0}^{\infty} 2\left[\sum_{l=0}^{p} a_p(l)x(n-l) - b(0)\delta(n) - b(0)\delta(n-n_0)\right]\left[\delta(n) + \delta(n-n_0)\right]$$

Thus,

$$x(0) - b(0) + \sum_{l=1}^{p} a_p(l)x(n_0-l) - b(0) = -x(n_0)$$

or, in vector form, we have

$$\mathbf{x}^T\mathbf{a} - 2b(0) = -x(0) - x(n_0)$$

Putting all of these together in matrix form yields

$$\begin{bmatrix} \mathbf{R}_x & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ -2b(0) \end{bmatrix} = -\begin{bmatrix} \mathbf{r}_x \\ x(0) + x(n_0) \end{bmatrix}$$

4. Hayes problem 4.25

Solution

(a) As we did in Example 4.7.1, we would like to find an ARMA(1,1) model for $x(n)$ that has the given autocorrelation values. Since the Yule-Walker equations are

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \\ r_x(2) & r_x(1) \end{bmatrix} \begin{bmatrix} 1 \\ a_1(1) \end{bmatrix} = \begin{bmatrix} c_1(0) \\ c_1(1) \\ 0 \end{bmatrix}$$

then the modified Yule-Walker equations for $a(1)$ are

$$r_x(1)a(1) = -r_x(2)$$

which gives $a_1(1) = -r_x(2)/r_x(1) = -1/2$.
For the moving average coefficients, we begin by computing $c_1(0)$ and $c_1(1)$ using the Yule-Walker equations as follows

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1(1) \end{bmatrix} = \begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix}$$

With the given values for $r_x(k)$, using $a_1(1) = -1/2$, we find

$$\begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix} = \begin{bmatrix} 3 & \frac{9}{4} \\ \frac{9}{4} & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 15/8 \\ 3/4 \end{bmatrix}$$

and

$$\big[C_1(z)\big]_+ = \tfrac{15}{8} + \tfrac{3}{4}z^{-1}$$

Multiplying by $A_1^*(1/z^*) = 1 - \frac{1}{2}z$ we have

$$\big[C_1(z)\big]_+ A_1^*(1/z^*) = \left(\tfrac{15}{8} + \tfrac{3}{4}z^{-1}\right)\left(1 - \tfrac{1}{2}z\right) = -\tfrac{15}{16}z + \tfrac{3}{2} + \tfrac{3}{4}z^{-1}$$

Therefore, the causal part of $P_y(z)$ is

$$\big[P_y(z)\big]_+ = \Big[\big[C_1(z)\big]_+ A_1^*(1/z^*)\Big]_+ = \tfrac{3}{2} + \tfrac{3}{4}z^{-1}$$

Using the symmetry of $P_y(z)$, we have

$$C_1(z)A_1^*(1/z^*) = B_1(z)B_1^*(1/z^*) = \tfrac{3}{4}z + \tfrac{3}{2} + \tfrac{3}{4}z^{-1}$$

Performing a spectral factorization gives

$$P_y(z) = B(z)B^*(1/z^*) = \frac{3}{4}(1 + z^{-1})(1 + z)$$

so the ARMA(1,1) model is

$$H(z) = \frac{\sqrt{3}}{2}\,\frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

(b) Yes. The model matches $r_x(k)$ for $k = 0, 1, 2$, and for $k > 2$ note that

$$r_x(k) = \tfrac{1}{2}r_x(k-1)$$

which they do.