EECS 225A Spring 2005

Homework 5 solutions

---

1. In the following, $Z(k) = R(k) + j \cdot I(k)$ is a zero-mean Gaussian random process (meaning $R(k)$ and $I(k-m)$ are zero-mean jointly Gaussian for all k and m) and wide-sense stationary with autocorrelation function $R_Z(m) = E[Z(k)Z^*(k-m)]$.
   a. In the real-valued case, we know that the statistics of a WSS random process are fully determined by the autocorrelation function. Investigate this in the complex case by trying to find the autocorrelation functions of $R(k)$ and $I(k)$ and their cross-correlation function in terms of $R_Z(k)$. In particular, show that the joint statistics of $R(k)$ and $I(k)$ are *not* uniquely determined by $R_Z(m)$.
   b. Define the *complementary autocorrelation function* as $\tilde{R}_Z(m) = E[Z(k)Z(k-m)]$. Show that $R_Z(m)$ and $\tilde{R}_Z(m)$ together uniquely determine the joint statistics of $R(k)$ and $I(k)$, and write down a set of equations that define that relationship.
   c. $Z(k)$ is said to be *circularly symmetric* if $\tilde{R}_Z(m) = 0$ for all m. What does $\tilde{R}_Z(0) = 0$ say about the joint distribution of $R(k)$ and $I(k)$ (for the same k)?
   d. What conditions guarantee that $R(k)$ and $I(k-m)$ are statistically independent for all m when the process is circular symmetric?
   e. Show that if a circularly symmetric process is applied to a linear time-invariant system, then circular symmetry is preserved at the output.

Solution

a. $R_Z(m)$ gives us two functions (its real and imaginary parts), whereas it takes three functions to fully specify the joint statistics of $R(k)$ and $I(k)$ (two autocorrelation functions and one cross-correlation function). So clearly the latter could not be uniquely specified by the former.

b.

$$2 \cdot R_r(k) = \mathrm{Re}\{R_Z(k)\} + \mathrm{Re}\{\tilde{R}_Z(k)\}$$
$$2 \cdot R_i(k) = \mathrm{Re}\{R_Z(k)\} - \mathrm{Re}\{\tilde{R}_Z(k)\}$$
$$2 \cdot R_{ri}(k) = -\mathrm{Im}\{R_Z(k)\} + \mathrm{Re}\{\tilde{R}_Z(k)\}$$

c.

$$2 \cdot R_r(k) = 2 \cdot R_i(k) = \mathrm{Re}\{R_Z(k)\}$$
$$2 \cdot R_{ri}(k) = -\mathrm{Im}\{R_Z(k)\}$$

The real and imaginary parts have the same variance $R_r(0) = R_i(0)$ and since $R_Z(0) = E|Z|^2$ must be real-valued, $2 \cdot R_{ri}(0) = -\mathrm{Im}\{R_Z(0)\} = 0$. Thus, the real and imaginary parts are identically distributed and statistically independent—the joint distribution is circularly symmetric. Also, the real and imaginary parts have identical autocorrelation functions, although their cross-correlation is not necessarily zero for non-zero lags.

d. We must have that $\mathrm{Im}\{R_Z(k)\} = 0$; that is, $R_Z(k)$ is real-valued. This does *not* imply that $Z(k)$ is real-valued ($R_i(k) \equiv 0$). In fact, $Z(k)$ *cannot* be real-valued and circularly symmetric at the same time, since $R_r(k) = R_i(k)$.

e. Let $V(k) = \sum_m Z(m) \cdot h(k-m)$. Then

$$E[V(k) \cdot V(n)] = E\left[\sum_m Z(m) \cdot h(k-m) \sum_i Z(i) \cdot h(i-n)\right]$$

$$E[V(k) \cdot V(n)] = \sum_m \sum_i E[Z(m) \cdot Z(i)] \cdot h(k-m) h(i-n) = 0$$

---

2. This example is drawn from digital communications. A received signal (represented as a vector in signal space) $\overrightarrow{X} \leftrightarrow \{x(t), -\infty < t < \infty\}$ is known to have finite energy, and to have embedded in it a signal of the form

$$\sum_{k=0}^{n-1} b_k \cdot \overrightarrow{H_k}$$

where $\overrightarrow{H_k} \leftrightarrow \{h(t-kT), -\infty < t < \infty\}$ is the time displacement (by kT) of a basic complex-valued waveform $\overrightarrow{H_0}$. The scalars b_k (called data symbols) are drawn from a finite alphabet of complex values and convey a stream of information bits. This is called *quadrature amplitude modulation* (QAM), and is commonly used in data modems.

   a. Our receiver design strategy is to consider all possible signals $\sum_{k=0}^{n-1} \hat{b}_k \cdot \overrightarrow{H_k}$ (for all possible combinations of $\{\hat{b}_k, 0 \le k \le n-1\}$) and choose the one with minimum distance from $\overrightarrow{X}$. Formulate this as a minimum-distance problem in Hilbert space.
   b. Show that the minimum-distance problem can be reformulated as minimizing distance in a new finite-dimensional Hilbert space. What is the appropriate inner product measure for this new space? HINT: You will want to 'complete the square'. A simple example of completing the square is $x^2 - bx = \left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4}$.
   c. Explore the implications of a minimum-phase factorization of the pulse autocorrelation function, as in $r_h(m) = \sigma^2 \cdot h_m \otimes h_{-m}^*$. Express the minimum distance in terms of h_k instead of $r_h(m)$ and show that the problem can be

reformulated as finding the minimum distance in a new countably infinite-dimensional Hilbert space. HINT: Following the approach described in class, define $u(k) = \ddot{b}_k \otimes h_k$ and $y(k) = \sigma^2 \cdot v(k) \otimes h^*_{-k}$, where $y(k)$ is the output of the sampled matched filter. Now express the minimum distance in terms of $u(k)$ and $v(k)$. Again, you will want to 'complete the square'.

Solution

a.

$$\min_{\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_{n-1}} D = \left\| \vec{X} - \sum_{k=0}^{n-1} \hat{b}_k \cdot \vec{H}_k \right\|^2 .$$

At first glance this looks like a projection problem; however, it is not, because the scalar coefficients are constrained to be valid data symbols chosen from a finite alphabet. Thus, the projection might be a reasonable first approximation, but is very unlikely to align with valid data symbols. Rather than applying the orthogonality principle, we should continue to work directly with the norm.

b. Writing this as

$$D = \left\| \vec{X} \right\|^2 + \left\| \sum_{k=0}^{n-1} \hat{b}_k \cdot \vec{H}_k \right\|^2 - 2 \cdot \mathrm{Re}\left\{ \sum_{k=0}^{n-1} \hat{b}_k^* \left\langle \vec{X} \middle| \vec{H}_k \right\rangle \right\} .$$

Defining

$$y_k = \left\langle \vec{X} \middle| \vec{H}_k \right\rangle \text{ (sampled output of a matched filter)}$$

$$r_h(m-n) = \left\langle \vec{H}_m \middle| \vec{H}_n \right\rangle \text{ (pulse autocorrelation function)}$$

this becomes

$$D = \left\| \vec{X} \right\|^2 + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \hat{b}_k \cdot r_h(k-i) \cdot b_i^* - 2 \cdot \mathrm{Re}\left\{ \sum_{k=0}^{n-1} \hat{b}_k^* \cdot y_k \right\}$$

The formulation suggests completing the square. Writing (for some yet-to-be-determined c_k)

$$\sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \left(\hat{b}_k - c_k \right) \cdot r_h(k-i) \cdot \left(\hat{b}_i - c_i \right)^* = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \hat{b}_k \cdot r_h(k-i) \cdot \hat{b}_i^* + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} c_k \cdot r_h(k-i) \cdot c_i^* - 2 \cdot \mathrm{Re}\left\{ \sum_{k=0}^{n-1} \hat{b}_k^* \cdot y_k \right\}$$

where

$$y_k = \sum_{i=0}^{n-1} r_h(i-k) \cdot c_i$$

and matching terms, we get

$$D = \left\| \vec{X} \right\|^2 + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \left(\hat{b}_k - c_k \right) \cdot r_h(k-i) \cdot \left(\hat{b}_i - c_i \right)^* - \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} c_k \cdot r_h(k-i) \cdot c_i^*$$

We recognize the second and third terms as Hermitian norms (as defined in lecture) over $\mathbb{C}^n$ of the form $\mathbf{x}^{\mathbf{T}} \mathbf{R} \mathbf{x}^*$ since the 'kernal' matrix $\mathbf{R}$ is Hermitian (these are Hermitian forms).

The first and third terms are not functions of the data symbols, and thus can be ignored. Thus, the data symbols can be determined by minimizing a Hermitian norm over $\mathbb{C}^n$. Unfortunately, there is no obvious way to avoid calculating this norm for all possible sequences of data symbols and choosing the sequence with the smallest norm. (In particular, the projection theorem is not helpful since the optimization is so constrained.) Before we do that, the c_k's must be determined by inverting the pulse autocorrelation matrix (or equivalently solving a set of linear equations).

c. Note that the 'isolated pulse' response to the matched filter followed by sampler is $s(k) = \left\langle \vec{H}_0 \middle| \vec{H}_k \right\rangle = r_h(-k)$. Since $s(k)$ has all the 'right' properties (Hermitian, positive-definite), cleaner results are obtained by performing a spectral factorization for $s(k)$, rather than $r_h(k)$ as suggested in the problem statement. (Note that if we had used this revised definition in lecture for the 'whitened matched filter' we would have gotten a discrete-time whitening filter $\dfrac{1}{H^*\left(\frac{1}{z^*}\right)}$ rather than $\dfrac{1}{H^*(z^*)}$, which looks cleaner.)

Per the problem hints:

$$s(m) = \sigma^2 \cdot h_m \otimes h^*_{-m} = \sigma^2 \cdot \sum_l h_l h^*_{l-m}$$

$$y(k) = \sigma^2 \cdot v(k) \otimes h^*_{-k} = \sigma^2 \cdot \sum_l v(l) \cdot h^*_{l-k}$$

$$u(k) = b_k \otimes h_k = \sum_l \hat{b}_l h_{k-l}$$

Returning to the problem formulation of a., we can attack the first term:

$$\sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \hat{b}_k \cdot r_h(k-i) \cdot b_i^* = \sigma^2 \cdot \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \hat{b}_k b_i^* \sum_l h_l h^*_{l+k-i} = \sigma^2 \cdot \sum_m \sum_{k=0}^{n-1} \hat{b}_k h_{m-k} \sum_{i=0}^{n-1} b_i^* h^*_{m-i}$$

$$\text{or} \quad \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \hat{b}_k \cdot r_h(k-i) \cdot b_i^* = \sigma^2 \cdot \sum_m |u(m)|^2$$

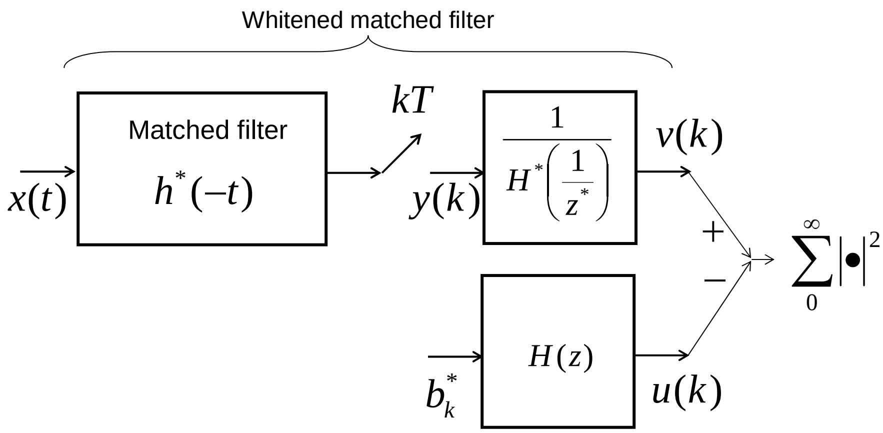
And the second term:

$$\mathrm{Re}\left\{\sum_{k=0}^{n-1} \hat{b}_k^* \cdot y(k)\right\} = \sigma^2 \cdot \mathrm{Re}\left\{\sum_{k=0}^{n-1} b_k^* \sum_l v(l) \cdot h^*_{l-k}\right\} = \sigma^2 \cdot \mathrm{Re}\left\{\sum_l v(l) \cdot u^*(l)\right\}$$

Combining these results, clearly the original minimization is equivalent to minimizing

$$\sum_l |u(l) - v(l)|^2$$

This solution is illustrated in the figure below. Note that this approach reduces the problem of finding minimum-distance data symbols to a discrete-time norm without the complication of the Hermitian kernel.


Whitened matched filter
Matched filter
$h^*(-t)$
$x(t)$
kT
$y(k)$
$\frac{1}{H^*\left(\frac{1}{z^*}\right)}$
$v(k)$
+
−
$\sum_0^\infty |\bullet|^2$
b_k^*
$H(z)$
$u(k)$