

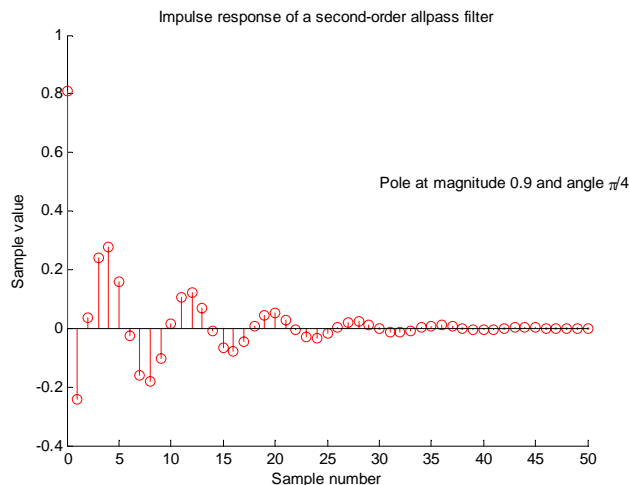
EECS 225A Spring 2005

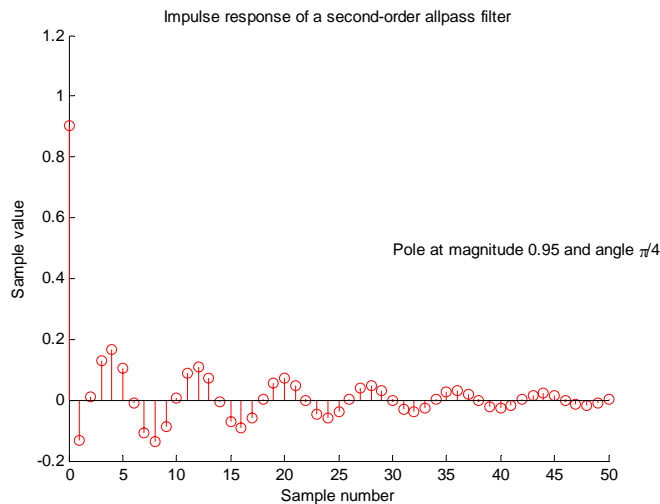
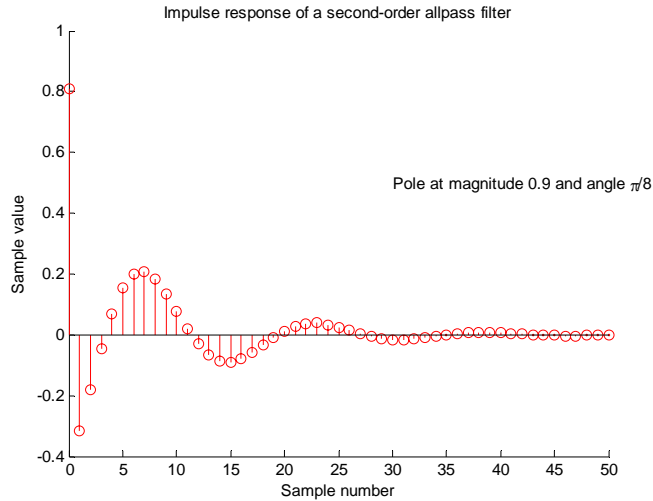
Homework 3 solutions

1. In lecture the unit-sample response of a second-order allpass filter was illustrated.
 - a. This filter has only a single independent parameter, which is the location of one of the two poles. Why?
 - b. Starting with the M-file used in class, experiment with changing both the radius and angle of the allpass filter pole. Explain intuitively the change in unit-sample response that you observe.

Solution

- a. The filter is allpass, so that the zero locations are predetermined by the pole locations (they are reflected through the unit circle). The filter was chosen to have a real-valued unit-sample response, so each pole location has a complex conjugate pole.
- b. Three cases are shown below. The first figure is the example given in class, modulus 0.9 and phase $\pi/4$. Intuitively we expect the frequency of ringing to be related to the angle of the pole. We can verify this by changing the angle from $\pi/4$ to $\pi/8$ and verify that the period of oscillation increases from 8 to 16 samples (second figure below). Intuitively we expect the modulus of the pole to be related to the rate of decay. We can verify this by changing the modulus from 0.9 to 0.95 and observe that the ringing decays more slowly (third figure below).





2. You are given a first-order stable and causal allpass filter with transfer function

$$\frac{c^* - z^{-1}}{1 - c \cdot z^{-1}}.$$

- What can you say about pole location c ?
- Show that if g_k is the input and h_k is the output of this filter, then the energy of g_k is more concentrated near zero-delay in the sense that

$$\sum_{k=0}^N |g_k|^2 \geq \sum_{k=0}^N |h_k|^2 \text{ for all } N > 0.$$

Thus, in this sense an allpass filter is dispersive. HINT: define a third signal f_k that is g_k filtered by the pole of the allpass filter only. Relate both g_k and h_k to f_k .

- Repeat b. for the (even simpler) stable and causal allpass filter z^{-n} .
- What do b. and c. say about minimum phase filters?

Solution

- a. $|c| < 1$ since the pole is at $z = c$ and the filter is said to be causal and stable.
- b. Since f_k is the output of a minimum-phase causal and stable filter (trivially minimum phase since it has no zeros), the inverse filter is causal and stable. Thus, we can recover g_k and h_k from f_k by the filters $1 - c \cdot z^{-1}$ and $c^* - z^{-1}$ respectively, or in the time domain

$$g_k = f_k - c \cdot f_{k-1} \quad \text{and} \quad h_k = c^* \cdot f_k - f_{k-1}.$$

By straightforward algebra, we find that

$$\sum_{k=0}^N |g_k|^2 - \sum_{k=0}^N |h_k|^2 = (1 - |c|^2) \cdot |f_N|^2 \geq 0.$$

- c. This is even more straightforward,

$$\sum_{k=0}^N |g_k|^2 - \sum_{k=0}^N |h_k|^2 = \begin{cases} \sum_{k=0}^N |g_k|^2, & N < n-1 \\ \sum_{k=N-n+1}^N |g_k|^2, & N \geq n-1 \end{cases}.$$

Clearly this is non-negative.

- d. Since a rational non-minimum phase filter can be decomposed as a rational minimum-phase filter followed by a rational allpass filter, among all rational transfer functions with the same magnitude response on the unit circle, the unit-sample response of a minimum-phase filter is the most concentrated near the origin (in the energy sense). That is, it has the least dispersion.

3. We are interested in rational transfer functions $H(z)$ that are stable and real-valued on the unit circle.
- Give an example of an $H(z)$ that is real-valued on the unit circle, but assumes *both* positive and negative values. What does this example teach you? Hint: Consider the simplest non-trivial rational $H(z)$ you can think of.
 - For the remainder of the problem, assume that $H(z)$ has *both* zeros and poles. What can you say about the pole and zero locations such that $H(z)$ will be causal? Non-causal? Neither?
 - What is the form of the *lowest-order* $H(z)$ (that is, how many poles and zeros does it have, and what constraints are there on location)?
 - Repeat c. for an $H(z)$ that has a *real-valued* unit-sample response.
 - Repeat c. for an $H(z)$ that is not only real-valued but *positive* real-valued on the unit circle.

Solution

- a. We know that $h(k) = h^*(-k)$ is all that required, and the simplest example would be

$$H(z) = a \cdot z + 1 + a \cdot z^{-1}$$

for a real valued. As expected, it is real-valued on the unit circle,

$$H(e^{j\omega}) = 1 + 2a \cdot \cos(\omega)$$

and this value is negative at some frequencies iff $|a| > \frac{1}{2}$. In this case, there are two zeros on the unit circle corresponding to $\cos(\omega) = -\frac{1}{2a}$. This occurs at two *different* frequencies (in fact, they are conjugate pairs since the unit-sample response is real). This teaches us that we can have zeros on the unit circle, and that they do not have to have multiplicity two (or more generally an even number). If they do not have even multiplicity, the frequency response will not be non-negative, as established on the last homework.

b. The statement that $H(z)$ is real-valued on the unit circle makes it implicit that the

ROC includes the unit circle. Also it implies that pole's and zeros come in $\left(z, \frac{1}{z^*}\right)$

(reflected through the unit circle) pairs, except perhaps for zeros on the unit circle. Since there are assumed to be poles, it follows that there must be poles both interior to and exterior to the unit circle, so $H(z)$ is two-sided (neither causal nor anti-causal).

In the case of zeros, any zero interior to the unit circle must be accompanied to a reflected zero outside the unit circle. It is also permissible for zeros to occur on the unit circle, and they need not come in pairs. We can also see that the unit-sample response must be two-sided in the time domain, since we must have

that $h(k) = h^*(-k)$.

c. We only need one pole inside the unit circle, and it must have a reflection outside the unit circle, so in total we have two poles. Similarly, zeros must come in pairs (one inside and one outside the unit circle or both on the unit circle).

d. Poles and zeros must come in (z, z^*) pairs, so there must be a minimum of two poles inside the unit circle. Adding the reflections through the unit circle, there will be (at minimum) four poles in total. For zeros, we only need two zeros if they are on the unit circle at complex-conjugate locations (as illustrated in part a.).

e. We must have two poles and two zeros. In this case, the two zeros can be in the form of a multiplicity-two zero on the unit circle, or alternatively a $\left(z, \frac{1}{z^*}\right)$ pair of zeros.

4. A one-bit analog-to-digital (A/D) converter is defined by a single threshold α and has two outputs, 0 and 1. Suppose a random variable X with probability density function $p_x(\cdot)$ is input to this one-bit A/D, and the output is defined as Y .
- What is the optimum (MMSE) estimator of X based on the observation of Y ?
 - The output of the A/D, Y , is input to a binary symmetric channel (BSC) with binary output Z characterized by the single parameter $0 \leq p \leq 1$ and defined by the conditional (transition) probability:

$$f_{z|y}(a|b) = \begin{cases} p, & a \neq b \\ 1-p, & a = b \end{cases}$$

What is the optimum (MMSE) estimator of X based on the observation of Z ?

- c. Interpret the results you got in b. for the special cases $p = 0$, $p = 1$, and $p = 0.5$.

Solution

a. This was derived in class. Let the MMSE estimator be defined by the two parameters

$$f(y) = \begin{cases} \mu_0, & y = 0 \\ \mu_1, & y = 1 \end{cases}$$

Then

$$\mu_0 = \frac{1}{p_0} \cdot \int_{-\infty}^{\alpha} x \cdot p_X(x) \cdot dx \quad \text{and} \quad \mu_1 = \frac{1}{1-p_0} \cdot \int_{\alpha}^{\infty} x \cdot p_X(x) \cdot dx$$

Where $p_0 = \int_{-\infty}^{\alpha} p_X(x) \cdot dx$.

b. The essential observation is that conditional on the observation of Y , X and Z are independent (please excuse the simplified notation):

$$p(x, z | y) = p(x | y) \cdot p(z | y).$$

The justification for this is that once you know Y , the randomness in X is due entirely to the input (nothing to do with Z), and the randomness in Z is due entirely to the BSC (nothing to do with X).

Aside: This assumption is equivalent to the condition stated in Homework #4 Problem #1, as you can see from:

$$p(z | x, y) = \frac{p(x, y, z)}{p(x, y)} = \frac{p(x, z | y)p(y)}{p(x | y)p(y)} = \frac{p(x | y)p(z | y)}{p(x | y)} = p(z | y).$$

Given this assumption, the trick is to write $p(x, y, z)$ in terms of transition probabilities that we know:

$p(x, y, z) = p(x, z | y)p(y) = p(z | y)p(x | y)p(y) = p(z | y)p(x, y) = p(z | y)p(y | x)p(x)$
Summing this over y ,

$$p(x, z) = p(x) \cdot \sum_y p(z | y)p(y | x) \quad \text{or} \quad p(x | z) = \frac{p(x) \cdot \sum_y p(z | y)p(y | x)}{p(z)}.$$

Recognizing that

$$p(y = 0 | x) = \begin{cases} 1, & x < \alpha \\ 0, & x \geq \alpha \end{cases} \quad \text{and} \quad p(y = 1 | x) = \begin{cases} 0, & x < \alpha \\ 1, & x \geq \alpha \end{cases}$$

we can perform the conditional-mean integral,

$$E[X | Z = z] = \int x \cdot p(x | z) \cdot dx = \frac{p(z | y = 0) \cdot p_0 \cdot \mu_0 + p(z | y = 1) \cdot (1 - p_0) \cdot \mu_1}{p(z)}$$

Finally,

$$p(z) = \sum_y p(z | y)p(y).$$

Thus the MMSE estimate corresponding to observation $Z = 0$ and $Z = 1$ are, respectively,

$$\tau_0 = E[X | Z = 0] = \frac{(1-p) \cdot p_0 \cdot \mu_0 + p \cdot (1-p_0) \cdot \mu_1}{(1-p) \cdot p_0 + p \cdot (1-p_0)}$$

$$\tau_1 = E[X | Z = 1] = \frac{p \cdot p_0 \cdot \mu_0 + (1-p) \cdot (1-p_0) \cdot \mu_1}{p \cdot p_0 + (1-p) \cdot (1-p_0)}$$

5. Do Hayes computer exercise C3.1.

Solution

See M-file hmwk03.m

a. If X is $U[0,1]$, then $\mu \cdot \log\left(\frac{1}{1-X}\right)$ will be exponentially distributed.

b. μ and μ^2

c. From Example 3.2.3 on p. 74, the variance of the sample mean is μ^2 / N . Letting $\varepsilon = \mu/100$ in Tchebycheff inequality on p. 73, we get that

$$\frac{\mu^2 / N}{\varepsilon^2} = \frac{(100)^2}{N} = \frac{1}{100} \text{ or } N = (100)^3 = 10^6.$$

This is a bound, not exact, but indicates that one million samples more than suffices.

c. This is mean $\mu = 2$.