EECS 225A Spring 2005

Homework 2 solutions

1. Consider a complex signal $\{z_k, 1 \le k \le n\}$. The goal is to find the best approximation to this signal in terms of a complex exponential with some fixed frequency ω; that is, the complex coefficients u and v such that $\{u \cdot e^{j\omega k} + v, 1 \le k \le n\}$ is the best approximation to z_k in the sense of minimizing the mean-square error $\mathrm{E} = \sum_{k=1}^{n} \left| z_k - u \cdot e^{j\omega k} - v \right|^2$.
   a. Find a set of sufficient conditions for a stationary point of E.
   b. Find an expression for the coefficients and resulting error suitable for computation.
   c. Compute the resulting coefficients numerically for signal $\{e^{j\omega k + j\pi/4} + x_k + jy_k, 1 \le k \le n\}$ where $\{x_k\}$ and $\{y_k\}$ are unit-variance real-valued white and independent Gaussian processes generated by a random number generator and $\omega = \pi/12, n = 1000$. Interpret the results. Repeat the calculation for a larger and smaller variance and interpret how they change.
   d. Repeat c. for signal $\{e^{j2\omega k + j\pi/4} + x_k + jy_k, 1 \le k \le n\}$. Note that there is now a mismatch between the signal frequency and the model frequency.

Solution

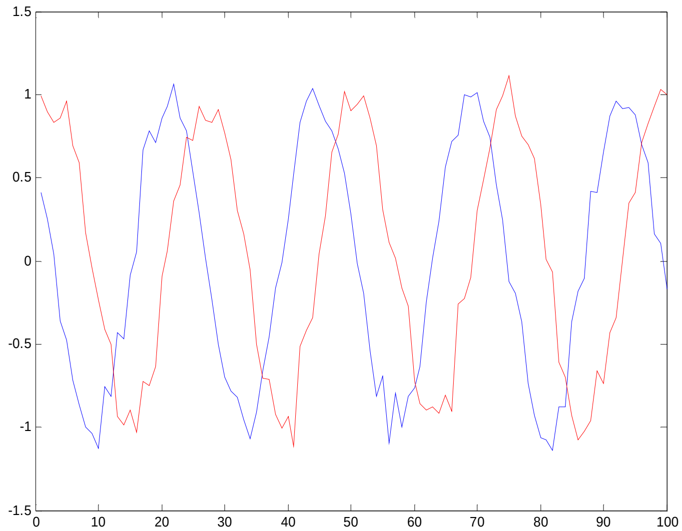
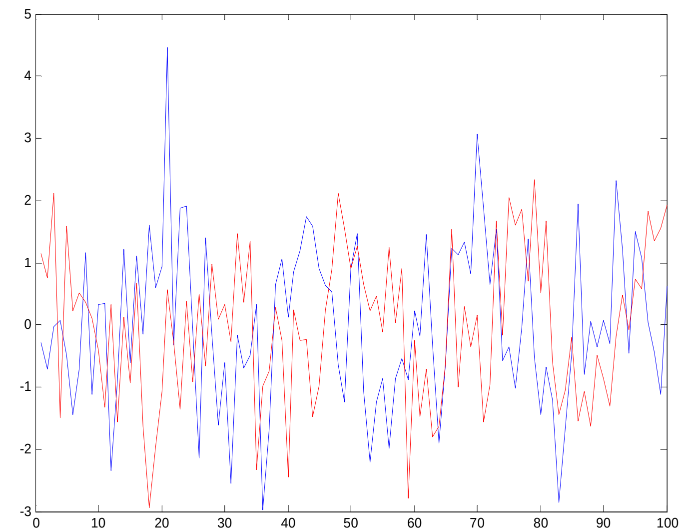
a. We can differentiate E w.r.t. u^* and v^* to obtain

$$n \cdot u + t^* \cdot v = r \text{ and } t \cdot u + n \cdot v = s \text{ where}$$

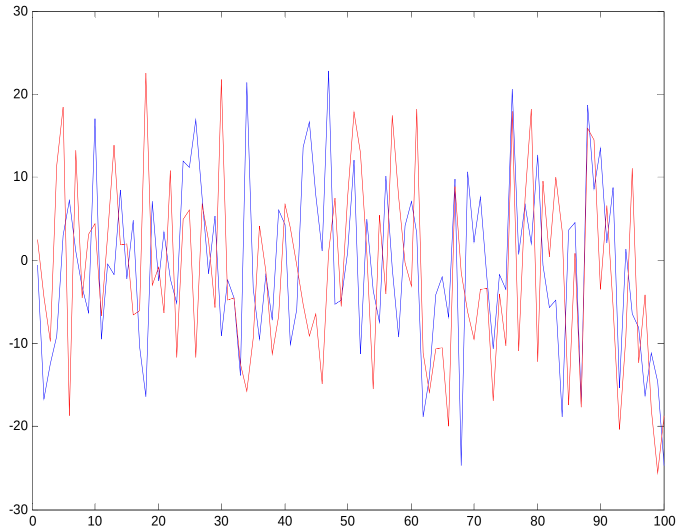
$$r = \sum_{k=1}^{n} z_k \cdot e^{-j\omega k} \text{ and } s = \sum_{k=1}^{n} z_k \text{ and } t = \frac{1 - e^{j\omega n}}{e^{-j\omega} - 1}$$

b. See M-file hmwk02.m

c. It is useful to look at the signal+noise for three cases:

Noise standard deviation	Signal, real and imaginary parts	Resulting *u* and *v*
0.1		$u \approx e^{j\pi/4}$ and $\lvert v \rvert \approx 0$ The model gives us an accurate characterization of the signal modulus and phase
1.0		The model remains pretty accurate; this is because of the opportunity to average out the noise over 1000 samples

10



The magnitude and phase estimate remain accurate within about 10-20%, but $|v|$ becomes pretty large (on the order of 0.2 to 0.4 due to the poor signal-to-noise ratio

d. All we have to change is the definition of r. The result is $|u| \approx 0$ with a widely varying angle, reflecting that the model is estimating the signal component at frequency $\pi/12$, which is in fact zero.

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2. Give an example of a rational transfer functions and associated ROC with each of the following properties. If no such rational transfer function exists, so state and give your reasoning.
   a. Causal real-valued unit-sample response.
   b. Anti-causal real-valued unit-sample response.
   c. Causal imaginary-valued unit-sample response.
   d. Anti-causal imaginary-valued unit-sample response.
   e. Causal unit-sample response that is neither pure real-valued nor pure imaginary valued.
   f. Real-valued unit-sample response that is two-sided.
   g. Real-valued on the unit circle, with both positive and negative values.
   h. Imaginary-valued on the unit circle.
   i. Non-negative real-valued on the unit circle.

Solution

a.

$$H(z) = (1-0.9e^{j\pi/4}\cdot z^{-1})\cdot(1-0.9e^{-j\pi/4}\cdot z^{-1}) \text{ with ROC = } |z|>0\text{, or}$$

$$H(z) = 1/((1-0.9e^{j\pi/4}\cdot z^{-1})\cdot(1-0.9e^{-j\pi/4}\cdot z^{-1})) \text{ with ROC = } |z|>0.9$$

These work because the zeros or poles come in complex-conjugate pairs. If we want the filter to be stable (nothing was said about this), it is important that the poles in the second case be inside the unit circle. Stating the ROC in addition to $H(z)$ is a key part of answering the question!

b. Again if we demand stability,

$$H(z)=1/((1-1.1e^{j\pi/4}\cdot z^{-1})\cdot(1-1.1e^{-j\pi/4}\cdot z^{-1})) \text{ with ROC = } |z|<1.1,$$

with poles outside the unit circle, works. The zeros case can be made anti-causal by reversing the exponent of z,

$$H(z)=(1-0.9e^{j\pi/4}\cdot z)\cdot(1-0.9e^{-j\pi/4}\cdot z) \text{ with ROC = } |z|<\infty.$$

c. Multiply any of the answers in a. by j.

d. Ditto for b.

e. Multiply any of the answers in a. by $(1+j)$.

f. If we want stability, then the ROC must include the unit circle. Real valued unit-sample response demands conjugate-pair poles and zeros. Examples would include

$$H(z)=(1-0.9e^{j\pi/4}\cdot z^{-1})\cdot(1-0.9e^{-j\pi/4}\cdot z^{-1})\cdot(1-2e^{j\pi/4}\cdot z^{-1})\cdot(1-2e^{-j\pi/4}\cdot z^{-1})$$

with ROC = $0<|z|<\infty$, or

$$H(z)=1/((1-0.9e^{j\pi/4}\cdot z^{-1})\cdot(1-0.9e^{-j\pi/4}\cdot z^{-1})\cdot(1-2e^{j\pi/4}\cdot z^{-1})\cdot(1-2e^{-j\pi/4}\cdot z^{-1}))$$

with ROC = $0.9<|z|<2$

g. Such a transfer function must be two-sided, since it requires that $H(e^{j\omega})=H^*(e^{j\omega})$ or $H(z)=H^*(\frac{1}{z^*})$, $h_k=h^*_{-k}$. An easy choice is $H(z)=G(z)+G^*(\frac{1}{z^*})$ since then $H(e^{j\omega})=2\cdot\mathrm{Re}\{G(e^{j\omega})\}$. Examples include:

$$H(z)=(1-c\cdot z^{-1})+(1-c^*\cdot z)$$

$$H(z)=\frac{1}{(1-c\cdot z^{-1})}+\frac{1}{(1-c^*\cdot z)}$$

In both cases the numerator polynomial is the same and evaluates on the unit circle to

$$1-c\cdot e^{-j\omega}+1-c^*\cdot e^{j\omega}=2-2\cdot\mathrm{Re}\{c\cdot e^{-j\omega}\}$$

This is real-valued as expected, but in order for this to change sign, we must have $|c|>1$.

Thus, finally, we have examples:

$$H(z)=(1-2e^{j\pi/4}\cdot z^{-1})+(1-2e^{-j\pi/4}\cdot z) \text{ with ROC = } 0<|z|<\infty, \text{ or}$$

$$H(z)=\frac{1}{(1-2e^{j\pi/4}\cdot z^{-1})}+\frac{1}{(1-2e^{-j\pi/4}\cdot z)} \text{ with ROC = } 0.5<|z|<2$$

h. Multiply any example in g. or i. by j.

i. Likewise the unit-sample response must be two-sided. An easy choice is $H(z)=G(z)\cdot G^*(\frac{1}{z^*})$ since then $H(e^{j\omega})=|G(e^{j\omega})|^2\geq 0$. Examples would include

$$H(z)=(1-0.5e^{j\pi/4}\cdot z^{-1})\cdot(1-0.5e^{-j\pi/4}\cdot z) \text{ with ROC = } 0<|z|<\infty, \text{ or}$$

$$H(z)=1/((1-0.5e^{j\pi/4}\cdot z^{-1})\cdot(1-0.5e^{-j\pi/4}\cdot z)) \text{ with ROC = } 0.5<|z|<2$$

3. Give an intuitive argument for the following statement. If you are feeling ambitious, offer a proof. For any rational transfer function that is non-negative real-valued on the unit circle, zeros on the unit circle must have even multiplicity (two, four, six, etc.).

Solution

An intuitive argument is that a single zero on the unit circle at angle θ will cause a phase shift of π as ω passes thru θ. Thus, unless other factors compensate (and they can't since they don't have a zero or pole at θ), a single zero will cause the overall transfer function to shift from positive to negative or vice versa. On the other hand, a zero of order $2n$ will cause a phase shift of $2\pi \cdot n$, which will not destroy the non-negative-real property.

Another intuitive argument can be had from the observation that poles or zeros off the unit circle must come in reflective pairs, giving factors like $(1-c \cdot z^{-1}) \cdot (1-c^* \cdot z)$. If we simply let $|c| = 1$, then for such a c we have that $c = 1/c^*$ and hence the zero-zero pairs on the unit circle coincide; that is, they come in multiplicity-two groups. Thus, pairs of zeros on the unit circle is a natural extension of the requirement for zero-zero pairs reflected through the unit circle. Two such groups gives us a multiplicity-four group, etc. Compare the frequency response (Z-transform on the unit circle) of a multiplicity-two group of zeros and a single zero on the unit circle:

$$(1-e^{j\theta} \cdot e^{-j\omega}) \cdot (1-e^{-j\theta} \cdot e^{j\omega}) = 2 - 2 \cdot \mathrm{Re}\left\{e^{j(\omega-\theta)}\right\} = 2 - 2 \cdot \cos(\omega-\theta)$$

$$(1-e^{j\theta} \cdot e^{-j\omega}) = 2j \cdot e^{-j(\omega-\theta)/2} \cdot \sin((\omega-\theta)/2)$$

While the first multiplicity-two factor is non-negative real, going to zero at the zero location $\omega = \theta$, the single-zero factor is not even real. Clearly what is happening is the two zeros in the first factor are canceling their phase shifts, and the sin() is being squared, making the resulting frequency response both real and non-negative. Intuitively there is no conceivable way that other factors (poles and zeros at other locations) can compensate for the non-zero phase shift of a single zero on the unit circle *at all frequencies*.