EE 225A Spring 2005

First Midterm Exam: Solutions

1. A function of a complex variable z is analytic in a region if (check one):

   - ☐ It is continuous at every point in the region.
   - ☺ It is differentiable with respect to z at every point in the region.
   - ☐ It is differentiable with respect to the real-part of z and the imaginary part of z at every point in the region.
   - ☐ All Cauchy sequences of points in the region converge to a point in the region.

2. Let $H(z)$ be of the form $H(z) = z^m \cdot \frac{B(z)}{A(z)}$ where $A(z)$ and $B(z)$ are respectively a p-th order and a q-th order polynomial in z. As you know, in general if you choose a different region of convergence (ROC), you get a different time function. Among the following choices, what types of time functions h_k are available (through the choice of an appropriate ROC) for *every* rational $H(z)$, including every possible value of *p*, *q*, and *m?* (You may need to check more than one. If you can identify a transfer function for which that type of time function is not available for any choice of ROC, then you should not check that item.)

   - ☐ A strictly causal time function ($h_k = 0$ for $k < 0$). (**Comment**: a counterexample is $H(z) = z$)
   - ☺ A loosely causal time function (there is an $N \geq 0$ such that $h_k = 0$ for $k < -N$). (**Comment**: Choose an ROC $|z| > D > 0$ where D is larger than the modulus of all poles. The only obstacle is poles at $z = \infty$; that is, $H(\infty) = \infty$ corresponding to positive powers of z. However, we can always write the transfer function in the form $H(z) = z^L \cdot H_1(z)$ such that $H_1(\infty) < \infty$. Then the ROC of $H_1(z)$ can be chosen for a strictly causal time response, and h_k becomes loosely causal.)
   - ☐ A strictly anticausal time function ($h_k = 0$ for $k > 0$).
   - ☺ A loosely anticausal (there is an $N \geq 0$ such that $h_k = 0$ for $k > N$).
   - ☐ A two sided (neither loosely causal nor loosely anticausal). (**Comment**: if all the poles have the same modulus, then there is no annulus ROC.)

☐ A time function with finite support (simultaneously loosely causal and loosely anticausal).

☐ None of the above options is available for every $H(z)$, m, p, and q.

3. Which of the following statements is true? (You may need to check more than one.)

   ☐ A rational transfer function corresponding to a real-valued signal must have, for every zero, a pole at its complex-conjugate location, and for every pole, a zero at its complex-conjugate location. (Reflection through the unit circle is defined as the function $1/z^*$).

   ☐ A rational transfer function that is real-valued on the unit circle must have, for every zero, a pole reflected through the unit circle, and for every pole, a zero reflected through the unit circle.

   ☺ The numerator and denominator polynomials of a rational transfer function corresponding to a real-valued unit-sample response must have real-valued coefficients.

   ☺ For a rational transfer function that is non-negative real-valued on the unit circle, any zeros on the unit circle must have even (multiple of two) multiplicity.

   ☐ None of the above statements is true.

4. Which of the following statements is true? (You may need to check more than one.)

   ☐ A causal and stable non-minimum phase rational transfer function can be turned into a causal and stable minimum-phase rational transfer function by cascading a causal and stable allpass transfer function. (**Comment**: this would require the allpass filter to have a pole outside the unit circle.)

   ☺ A causal and stable non-minimum phase rational transfer function can be turned into a causal and stable minimum-phase transfer function by cascading an anticausal and stable allpass transfer function.

   ☺ Every non-minimum phase causal and stable rational transfer function can expressed as a cascade of a causal and stable minimum-phase transfer function and a causal and stable allpass transfer function.

   ☺ Every causal and stable allpass rational transfer function is non-minimum phase.

☐ Every anticausal and stable allpass rational transfer function is minimum-phase. (**Comment**: allpass filters must have pole-zero pairs reflected through the unit circle, so they are never minimum-phase—either a pole or a zero must be outside the unit circle.)

5. Only rational transfer functions can be minimum-phase. This statement is (check one):

   ☐ True

   ☺ False

6. You want to form a minimum mean-square error estimate of a random variable X based on the observation of a random variable Y. Which of the following are true statements? (Check all that apply, you may have to check more than one.)

   ☐ The optimum estimate is always linear, of the form $a \cdot Y$.

   ☐ The optimum estimate is never linear.

   ☺ Even where a linear estimate is not optimum, it can still be useful because it is simple and analytically tractable.

   ☐ To determine the optimum estimate, we must know the joint probability density of X and Y, $p_{XY}(x, y)$ (or equivalently the joint distribution function). (**Comment**: All we need know is $p_{X|Y}(x \,|\, Y = y)$, from which $p_{XY}(x, y)$ cannot be inferred without additional information.)

   ☐ To determine the optimum estimate, we need only know all the second order statistics of X and Y (means, variances, covariance).

7. In modeling a deterministic signal as the unit-sample response of a rational transfer function, the most common procedure is which of the following? (Check the one that applies.)

   ☺ Solve a set of linear equations to determine the coefficients of the denominator polynomial, and then solve another set of linear equations to determine the coefficients of the numerator polynomial.

   ☐ Solve a set of linear equations to determine the roots of the denominator polynomial, and then solve another set of linear equations to determine the roots of the numerator polynomial.

☐ Solve a set of linear equations to determine the coefficients of the numerator polynomial, and then solve another set of linear equations to determine the coefficients of the denominator polynomial.

☐ Solve a set of linear equations to determine the roots of the numerator polynomial, and then solve another set of linear equations to determine the roots of the denominator polynomial.

8. For a system of linear equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is the unknown vector and is to be determined, which of the following statements are true? (Check all that apply, you may have to check more than one)

(**Comment**: Here is a table that summarizes the possibilities:)

Case	**Rank**	**Unique solution**	**No solution**	**Multiple solutions**
Square rows = columns # variables = # equations	Full	$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$		
	Deficient (non-zero null space)		If $\mathbf{b}$ is not in column space	If $\mathbf{b}$ is in column space; add any vector in null space
Overdetermined rows > columns # variables < # equations	Full	If $\mathbf{b}$ is in column space	If $\mathbf{b}$ is not in column space	
	Deficient (non-zero null space)		If $\mathbf{b}$ is not in column space	If $\mathbf{b}$ is in column space; add any vector in null space
Underdetermined rows < columns # variables > # equations (always non-zero null space)	Full			An $\mathbf{x}$ in the row space must match constraints; add any vector in null space
	Deficient		If no $\mathbf{x}$ in the row space matches constraints	If an $\mathbf{x}$ in the row space matches constraints; add any vector in null space

☺ If the number of unknown variables equals the number of linear equations, there are three possible cases: there is a unique solution, there is no solution, or there are multiple solutions.

☺ If the number of unknown variables is larger than the number of linear equations, there are two possible cases: there is no solution, or there are multiple solutions.

☐ If the number of unknown variables is smaller than the number of linear equations, there are two possible cases: there is a unique solution, or there are multiple solutions.

☐ If the number of unknown variables is larger than the number of linear equations and the matrix $\mathbf{A}$ is full rank, there are two possible cases: there is no solution, or there are multiple solutions.

☐ If the number of unknown variables is smaller than the number of linear equations and the matrix $\mathbf{A}$ is full rank, there are two possible cases: there is a unique solution, or there are multiple solutions.

9. Suppose a lattice filter is designed to be the optimum finite-order linear predictor of a wide-sense stationary random process. Check all the following statements that are true (you may need to check more than one).

☐ The reflection coefficients are all less than unity in modulus.

☺ The reflection coefficients are all less than or equal to unity in modulus.

☐ A reflection coefficient that is larger than unity indicates that the autocorrelation function of the process is only positive semi-definite, not strictly positive definite.

☐ No reflection coefficient is ever zero.

☺ When a reflection coefficient is zero, the mean-square prediction error does not decrease through that stage of the filter.

☺ When a reflection coefficient is smaller in magnitude, that indicates that this stage of the filter is less effective in further reducing the mean-square prediction error than when this same reflection coefficient is larger in magnitude.

10. Suppose a lattice filter is designed to be the optimum finite-order linear predictor of a wide-sense stationary random process. Check all the following statements that are true (you may need to check more than one).

☐ The forward prediction error $E_n(k)$ (order n at time k) is uncorrelated with $E_{n-1}(k)$

☐ $E_n(k)$ is uncorrelated with $E_n^R(k-n)$ (the reverse or backward prediction error of order n at time $k-n$).

☺ $E_{n-1}^R(k-n)$ is uncorrelated with $E_n(k)$

☺ $\left\|\vec{E}_{n-2}^R(k-n+2)\right\| = \left\|\vec{E}_{n-2}(k-2)\right\|$

☺ $E_{n-2}(k-2)$ is uncorrelated with $E_{n-4}^R(k-n)$

☐ None of these is correct.

11. Given a rational function $H(z)$, z is allowed to travel counterclockwise around the circle $|z|=1$ and subsequently counterclockwise around the circle $|z|=2$. It is observed that the trajectory of $H(z)$ traverses about the origin 3 times in the clockwise direction in the first case and 2 times in the counterclockwise direction in the second case. From the functional form of $H(z)$ we can see directly that it has no poles or zeros at the origin. Which of the following conclusions are consistent with these observations? (More than one may be consistent, check all that apply.)

☐ $H(z)$ is minimum phase

☺ In the annulus $1<|z|\leq 2$, $H(z)$ has 7 zeros and 2 poles.

☺ In the disk $|z|\leq 1$, $H(z)$ has 3 zeros

☐ In the disk $|z|\leq 2$, $H(z)$ has 2 poles (**Comment**: Since it must have at least 3 poles inside the unit circle, it must have at least 3 poles in this disk.)

☐ In the region $|z|\geq 2$, $H(z)$ has no poles or zeros

☐ None of the above is consistent with the observations

12. Which of the following statements apply to an *autoregressive* wide-sense stationary random process? (Check all that apply, more than one may apply.)

☐ It is also strictly stationary.

☺ It's power spectrum is rational, and every pole is accompanied by another pole reflected through the unit circle. (**Comment**: Statement is not complete, since it doesn't mention the fact that there can be no zeros. Nevertheless it is true as far as it goes.)

☺ Its power spectrum is rational, and it has no zeros.

- ☺ The minimum mean-square error backward or reverse predictor of order ∞ is actually a finite-order predictor. (**Comment**: True of a forward predictor, so must also be true of a reverse predictor.)
- ☐ It must be a Gaussian process.
- ☺ Its autocorrelation function must obey a difference or recurrence relationship.
- ☐ None of the above is correct.

1. A possibly complex-valued random process $U(k) = X(k) + jY(k)$ is zero-mean and wide-sense stationary, and is known to have autocorrelation function

$$r_U(m) = \exp\{-\alpha \cdot |m|\}$$

for some real-valued constant $\alpha > 0$.

a. What are the strongest *general* assertions you can make about the joint statistics of $X(k)$ and $Y(k)$? To get full credit, your assertions must be exhaustive, and you should state all you can infer about both the autocorrelation and power spectrum aspects of their joint statistics.

b. Is the random process $V(k) = U(k) \cdot \exp\{j\omega k\}$, for some real-valued constant $-\pi < \omega \le \pi$, wide-sense stationary? Justify your answer.

c. What are the coefficients of the minimum-mean-square-error ∞-order linear predictor of $U(k)$ based on its past?

d. Consider the minimum-mean-square-error linear estimate of $U(k)$ based on $U(k-N-m),\ m \ge 0$ for some given $N \ge 1$. Find the coefficients of this estimate.

Solution

a. Strictly speaking, we can't assume that $X(k)$ and $Y(k)$ are jointly WSS, but have to show it. In fact, we showed in a homework earlier that the autocorrelation function of a complex-valued process does not fully specify the joint statistics of the real and imaginary parts. In fact, that $U(k)$ is WSS does not guarantee that $X(k)$ and $Y(k)$ are jointly WSS.

In practice, I gave full credit if you assumed they were WSS and went from there. (They *can* be, and are likely to be, it is just that they don't *have* to be.) Making that assumption, by direct calculation,

$$r_U(m) = r_X(m) + r_Y(m) + j \cdot (r_{XY}(-m) - r_{XY}(m)).$$

Thus we get

$$r_X(m) + r_Y(m) = e^{-\alpha|m|} \text{ or } P_X(z) + P_Y(z) = P_U(z)$$
$$r_{XY}(m) = r_{XY}(-m) \text{ or } P_{XY}(z) = P_{XY}(z^{-1})$$

- Both the real and imaginary parts are zero-mean, like $U(k)$.
- The process $U(k)$ may or may not be real-valued.

- The cross-correlation function must be even, and thus the cross-power spectrum on the unit circle must be symmetric about $\omega = 0$, $P_{XY}(e^{j\omega}) = P_{XY}(e^{-j\omega})$.

b. By direct calculation of the autocorrelation function,

$$E[V(k)\cdot V^*(k-m)] = E[U(k)\cdot e^{j\omega k}\cdot U^*(k-m)\cdot e^{-j\omega(k-m)}] = r_U(m)\cdot e^{j\omega m}$$

This not being a function of *k*, $V(k)$ is WSS.

c. The power spectrum will have a single pair of poles, and likely does not have any zeros, so it is autoregressive of order $p = 1$. Thus we expect that the optimum predictor will be a first-order predictor. From the results of d. for $N = 1$, the single coefficient is $a_1(1) = -e^{-\alpha}$ and the prediction error filter is

$$A_\infty(z) = A_1(z) = 1 - e^{-\alpha} z^{-1}.$$

Method 1: Realizing that the optimum predictor must be first order, find the coefficient and prove that it is optimal using the orthogonality principle, as we did in d. below.

Method 2: Find the power spectrum

$$P_U(z) = \frac{1 - e^{-2\alpha}}{(1 - e^{-\alpha} z^{-1})(1 - e^{-\alpha} z)}$$

and then observe that the filter $A_\infty(z) = A_1(z) = 1 - e^{-\alpha} z^{-1}$ is a whitening filter and thus must be an optimum ∞-order prediction error filter.

d. Given the speculation of c. for $N = 1$, it seems likely that a first order predictor (based on only the most recent sample) will do.

Method 1: Invoke the orthogonality principle to see if we might be that lucky. Let β be the coefficient of a first order predictor based on the most recent sample, so that

$$E[(U(k) - \beta\cdot U(k-N))U^*(k-N)] = 0 \text{ or } \beta = \frac{r_U(N)}{r_U(0)} = e^{-\alpha N}.$$

Now we can check to see if this happens to be the optimum ∞-order predictor by checking if the orthogonality principle is satisfied,

$$E[(U(k) - \beta\cdot U(k-N))U^*(k-N-m)] = r_U(N+m) - e^{-\alpha N}\cdot r_U(m) = 0 \text{ for } m \ge 0.$$

We lucked out! The first-order predictor is optimum.

Method 2: Without making any bold guesses as to the solution, find the appropriate set of normal equations and solve them. From the orthogonality principle,

$$\left\langle U(k) + \sum_{i=0}^{\infty} a(i) \cdot U(k-N-i) \middle| U(k-N-m) \right\rangle = 0, \ m \geq 0$$

$$r_U(m+N) = e^{-N\alpha} r_U(m) = \sum_{i=0}^{\infty} a(i) \cdot r_U(m-i), \ m \geq 0$$

This can be rewritten as

$$e^{-N\alpha} r_U(m) = a(m) \otimes r_U(m), \ m \geq 0$$

By inspection, a solution to these equations is $a(m) = e^{-N\alpha}\delta(m)$. The projection theorem tells us this solution must be unique.

2. Given stable and causal unit-sample responses $g(k)$ and $d(k)$, suppose the least-squares (LS) FIR least-squares inverse filter $h_N(k), \ 0 \leq k \leq N-1$ is chosen to minimize

$$\varepsilon = \sum_{k=0}^{\infty} |e(k)|^2 \ \text{ where } \ e(k) = d(k) - g(k) \otimes h_N(k) .$$

Let $E(z)$ be the z-transform of $e(k)$, then by Parseval's relation,

$$\varepsilon = \sum_{k=0}^{\infty} |e(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| E(e^{j\omega}) \right|^2 \cdot d\omega .$$

In a particular application, you want more control over the accuracy of the LS inverse filter as a function of frequency (with greater accuracy in some frequency bands than others) so you define a positive real-valued weighting function $W(e^{j\omega}) > 0$ and reformulate the criterion for design of the inverse filter as minimizing

$$\varepsilon_W = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| E(e^{j\omega}) \right|^2 W(e^{j\omega}) \cdot d\omega .$$

a. Find a set of N linear equations which, when solved, would minimize ε_W. These equations may be expressed in terms of discrete-time convolutions and sums, but must not contain any un-evaluated integrals.

b. State conditions under which the equations you obtained have a unique solution.

Solution

a. Several methods will work here, and some are more difficult to carry out.

Method 1: Since $W(e^{j\omega}) > 0$ it admits the spectral factorization $W(z) = \sigma^2 \cdot S(z) S^*\left(\frac{1}{z^*}\right)$ where $S(z)$ is monic, stable and causal, and minimum-phase. Then the criterion can be restated as minimizing

$$\varepsilon_W = \sigma^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| S(e^{j\omega}) E(e^{j\omega}) \right|^2 \cdot d\omega .$$

Define $E'(z) = S(z)E(z)$, and then

$$e'(k) = e(k) \otimes s(k) = (d(k) \otimes s(k)) - h_N(k) \otimes (g(k) \otimes s(k)) .$$

Thus, the previous solution can be repeated, with $d(k)$ replaced by $u(k) = d(k) \otimes s(k)$ and $g(k)$ replaced by $v(k) = g(k) \otimes s(k)$. Defining

$$r_g(m) = \sum_{k=0}^{\infty} g(k) g^*(k-m)$$
$$r_{dg}(m) = \sum_{k=0}^{\infty} d(k) g^*(k-m)$$

The set of linear equations to be solved for the original problem is

$$r_{dg}(j) = \sum_{i=0}^{N-1} h_N(i) \cdot r_g(j-i), \; 0 \le j \le N-1 .$$

With the new formulation of the problem all we have to do is replace the two correlation functions,

$$r_{uv}(j) = \sum_{i=0}^{N-1} h_N(i) \cdot r_v(j-i), \; 0 \le j < N-1 .$$

Method 2: Several students used $\sqrt{W(z)}$ in place of Method 1's $S(z)$. This may work, but you have to realize that $\sqrt{W(z)}$ is not an analytic function and that the time sequence will not be causal (it is a Hermitian time sequence). The whole idea with spectral factorization is to choose a phase response corresponding to magnitude response $\sqrt{W(z)}$ that gives you a causal time sequence, greatly simplifying the remainder of the analysis.

Method 3: Several students worked with $W(z)$ directly rather than factoring it. This will work if you recognize that

$$E(z)E^*\left(\frac{1}{z^*}\right)W(z) \leftrightarrow e(k) \otimes e^*(-k) \otimes w(k)$$

and also recognize that $w(k) = w^*(-k)$ (the time weighting function is not causal). Working this out you get an infinite-dimensional Hermitian form. You can then minimize, but it is messier than Method 1 (which reverts to a previous solution).

Method 4: Several students worked with the integral directly, differentiating it. This will work if done with care, but has the same limitation as Method 3.

b. Method 1: In the original problem the solution is unique iff the N-th order autocorrelation matrix is positive definite, which is true iff the vectors $\vec{G}_m \leftrightarrow g(k-m), k \geq 0$ are linearly independent for $0 \leq m \leq N-1$. Since $\vec{G}_m$ is a stationary signal, these vectors are linearly dependent iff an *m*-th order predictor yields zero error for $m \leq n$. We know that this happens if the power spectrum consists of a finite set of impulses; that is, the autocorrelation function $\left\langle \vec{G}_k \middle| \vec{G}_{k-m} \right\rangle$ consists of a superposition of a finite set of complex exponentials.

Since the power spectrum of $v(k)$ is the power spectrum of $g(k)$ multiplied by $S(z)S^*\left(\frac{1}{z^*}\right) = W(z)$, if we choose a well-behaved (e.g. positive and continuous) weighting function it will not affect the form of the power spectrum (consisting of a superposition of impulses or not), and thus will not affect the uniqueness of the solution one way or another.