

# EECS 225A Spring 2005

## Matrix inversion lemma

David G Messerschmitt

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When both  $\mathbf{A}$  and  $(\mathbf{A} + \mathbf{u}\mathbf{v}^H)$  are invertible (where  $\mathbf{A}$  is a square matrix and  $\mathbf{u}$  and  $\mathbf{v}$  are column vectors), the matrix inversion lemma states that

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{\mathbf{1} + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}}$$

This identity is widely used in signal processing and control, because it allows us to invert a rank-one extension to a non-singular matrix without requiring a matrix inversion.

In class I derived this using the spectral decomposition theorem for a Hermitian matrix. Mesmerized by obtaining the right answer, I neglected to verify that the matrix  $\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H$  is actually Hermitian (which it is not in general).

The following simple argument verifies the identity, although it unfortunately gives absolutely no hint as to where it comes from. Multiplying the matrix by its supposed inverse, the result should be the identity matrix:

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^H) \cdot \left( \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{\mathbf{1} + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}} \right) = \mathbf{I} + \mathbf{u}\mathbf{v}^H\mathbf{A}^{-1} - \frac{\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{\mathbf{1} + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}}$$

Recognizing that  $\mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}$  is a scalar and thus commutes with everything, this product becomes

$$\mathbf{I} + \mathbf{u}\mathbf{v}^H\mathbf{A}^{-1} - \frac{\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1} + (\mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}) \cdot \mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{\mathbf{1} + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}} = \mathbf{I}.$$