Nonlinear Controller Design for the HVAC system of Energy Efficient Buildings

Course project report

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Abstract

This report focuses on modeling the thermal behavior of buildings and designing a nonlinear control for their HVAC systems. The problem of developing a good model to capture the heat storage and heat transmission properties of building thermal elements such as rooms and walls is addressed by using the lumped capacitance method. The nonlinear equations governing the system dynamics are derived using the thermal circuit approach. First we have approached the problem by linearizing the system about an equilibrium point, which yields a linear model. Then we have implemented some nonlinear control techniques in order to investigate the importance and effectiveness of the nonlinearities of the system. The controllability and observability analysis are done using Lie bracket and Lie derivative operations in the next section. We have implemented input-output linearization and input-state linearization techniques to transform the original system model into equivalent models of simpler forms. This approach is completely different from the conventional Jacobian linearization technique, because feedback linearization is achieved by exact state transformation and feedback, rather than by linear approximations of the dynamics. Theorem of Frobenius is used to determine the feasibility of input-state linearization. Sliding mode control is another control technique that was used to design a robust controller for the system. The simulation results are brought at the end and the performance of different controllers are compared.
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Chapter 1

Introduction

This report focuses on the controller design for the Heating Ventilating and Air Conditioning (HVAC) systems of commercial, residential and industrial buildings. In 2001, building heating ventilation and air-conditioning (HVAC) systems accounted for approximately 30% of total energy consumption in the United States. This is greater than transportation, which accounted for approximately 28% of total energy consumption. However, the energy consumed by HVAC systems is less evident and distributed across residential, commercial and industrial sectors. HVAC systems, in particular cooling, are one of the fastest growing energy consumers in the United States. This trend started in the 1970s, and continues today. However, much of this growth has been offset by gains in efficiency. There is still much room for improvement in the efficiency of such systems with technology that already exists such as designing robust and smart controllers for these systems.
Chapter 2

Mathematical Model

The first step in designing a control system for a given physical plant is to derive a meaningful model of the plant, i.e., a model that captures the key dynamics of the plant in the operational range of interest. Modeling is basically the process of constructing a mathematical description (usually a set of differential equations) for the physical system to be controlled. Two points can be made about modeling. First, one should use good understanding of the system dynamics and the control tasks to obtain a tractable yet accurate model for control design. Note that more accurate models are not always better, because they may require unnecessarily complex control design and analysis and more demanding computation. The key here is to keep “essential” effects and discard insignificant effects in the system dynamics in the operating range of interest. Second, modeling is more than obtaining a nominal model for the physical system; it should also provide some characterization of the model uncertainties, which may be used for robust design, adaptive design, or merely simulation[2].

In this project the thermal behavior of a one room building is studied. The state space representation of the system can be derived by applying the lumped capacitance method and using the “Nodal Analysis” on the proposed circuit model to derive the differential equations governing the thermal behavior of the building. A layout of the building is presented below:
Using the *lumped capacitance method* we can model the thermal behavior of the walls and the air in the rooms, using thermal capacitance and thermal resistance of the air and the walls. This modeling method is illustrated in the figure below:

![Figure 2.1: simple three-room building](image)

![Figure 2.2: Heat interchange through exterior and interior walls.](image)

for wall 1 we can write:

$$\frac{T_{\infty} - T_{w1}}{(R_{co})_{1}} + \frac{T_1 - T_{w1}}{(R_{ci})_{1}} + \alpha q_{rad1} = C_{w1} \frac{dT_{w1}}{dt} \quad (2.1)$$

Where $R_{ci}$ and $R_{co}$ for each wall are defined to be
\[ R_{ci} = R_{cv_i} + \frac{R_w}{2} \]

\[ R_{co} = R_{cv_o} + \frac{R_w}{2} \]

And \( T_\infty \) is the outside temperature, \( T_{w_i} \) is the temperature of wall \( i \), \( T_1 \) is the temperature of room, \( R_{cv_o} \) is the outside thermal resistance due to convective heat transfer, and is defined as \( R_{cv_o} = \frac{1}{h_o A_i} \) where \( h_o \) is the convective heat transfer coefficient and \( A_i \) is the area of wall \( i \), \( C_{w_i} \) is the specific heat capacity of the wall \( i \), \( \alpha \) is the absorptivity coefficient of walls, and \( q_{rad_i} \) is the radiation heat transfer to wall \( i \). Similar equation can be written for walls number 2, 3 and 4:

\[ \frac{T_\infty - T_{w_2}}{(R_{co})_2} + \frac{T_1 - T_{w_2}}{(R_{ci})_2} + \alpha q_{rad_2} = C_{w_2} \frac{dT_{w_2}}{dt} \quad (2.2) \]

\[ \frac{T_\infty - T_{w_3}}{(R_{co})_3} + \frac{T_1 - T_{w_3}}{(R_{ci})_3} + \alpha q_{rad_3} = C_{w_3} \frac{dT_{w_3}}{dt} \quad (2.3) \]

\[ \frac{T_\infty - T_{w_4}}{(R_{co})_4} + \frac{T_1 - T_{w_4}}{(R_{ci})_4} + \alpha q_{rad_4} = C_{w_4} \frac{dT_{w_4}}{dt} \quad (2.4) \]

By doing the same \textit{Nodal Circuit Analysis} for each room in the building we can get the following equations:

\[ \frac{T_{w_1} - T_1}{(R_{ci})_1} + \frac{T_{w_2} - T_1}{(R_{ci})_2} + \frac{T_{w_3} - T_1}{(R_{ci})_3} + \frac{T_{w_4} - T_1}{(R_{ci})_4} + \dot{m}_1 c_p (T_0 - T_1) + q_{int} = C_{r_1} \frac{dT_1}{dt} \quad (2.5) \]

Where \( \dot{m}_1 \) is the air mass flow through the ducts into room number 1, \( c_p \) is the specific heat of air, \( T_0 \) is the temperature of the chilled air or hot air that comes into the rooms through the ducts, and \( q_{int} \) is the heat generation inside the rooms which can be from electrical devices such as computers, or from humans, lighting and etc.
Below we have the dynamic equations of the system.

\[
\dot{x} = f(x) + g(x).u + d(t) \tag{2.6}
\]

where $x$, $f(x)$, $g(x)$ and the disturbance vector $d(t)$ are as follows:

\[
x = [T_1 \ T_{w1} \ T_{w2} \ T_{w3} \ T_{w4}]^T \tag{2.7}
\]

\[
f(x) = A.x = \begin{bmatrix}
                 \frac{-1}{C_r} \sum_{j=1}^{4} \frac{1}{R_{cij}} & \frac{1}{C_r R_{ci1}} & \frac{1}{C_r R_{ci2}} & \frac{1}{C_r R_{ci3}} & \frac{1}{C_r R_{ci4}} \\
                 \frac{1}{C_w1 R_{wi1}} & \frac{-1}{R_i C_{w1}} & 0 & 0 & 0 \\
                 \frac{1}{C_w2 R_{wi2}} & 0 & \frac{-1}{R_i C_{w2}} & 0 & 0 \\
                 \frac{1}{C_w3 R_{wi3}} & 0 & 0 & \frac{-1}{R_i C_{w3}} & 0 \\
                 \frac{1}{C_w4 R_{wi4}} & 0 & 0 & 0 & \frac{-1}{R_i C_{w4}} \\
\end{bmatrix} \cdot x
\]

Where $R$ is defined as

\[
\frac{1}{R} = \frac{1}{R_{ci}} + \frac{1}{R_{co}}
\]

and $g(x)$ is defined as

\[
g(x) = \begin{bmatrix}
                 \frac{C_p}{C_r} (T_0 - T_1) \\
                 0 \\
                 0 \\
                 0 \\
\end{bmatrix}
\]

and the input $u$ is the mass air flow into the room, i.e. $\dot{m}_1$. 

The disturbance vector is also defined as:

$$d(t) = \begin{bmatrix}
\frac{1}{C_r} q_{int} \\
\frac{T_{\infty}}{C_{w_1} R_{c_1}} \frac{\alpha}{C_{w_1}} q_{rad_1} \\
\frac{T_{\infty}}{C_{w_2} R_{c_2}} \frac{\alpha}{C_{w_2}} q_{rad_2} \\
\frac{T_{\infty}}{C_{w_3} R_{c_3}} \frac{\alpha}{C_{w_3}} q_{rad_3} \\
\frac{T_{\infty}}{C_{w_4} R_{c_4}} \frac{\alpha}{C_{w_4}} q_{rad_4}
\end{bmatrix}$$
Chapter 3

Stability Analysis

Given a control system, the first and most important question about its various properties is whether it is stable, because an unstable control system is typically useless and potentially dangerous. Qualitatively, a system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after. The motions of a pendulum starting near its two equilibrium points, namely, the vertical up and down positions, are frequently used to illustrate unstable and stable behavior of a dynamic system. Every control system, whether linear or nonlinear, involves a stability problem which should be carefully studied[2].

The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19th century by the Russian mathematician Alexandr Mikhailovich Lyapunov. Lyapunov’s work, The General Problem of Motion Stability, includes two methods for stability analysis (the so-called linearization method and direct method) and was first published in 1892. The linearization method draws conclusions about a nonlinear system’s local stability around an equilibrium point from the stability properties of its linear approximation. The direct method is not restricted to local motion, and determines the stability properties of a nonlinear system by constructing a scalar “energy-like” function for the system and examining the function’s time variation. Here we will study the local stability of the system using Lyapunov’s direct method[2].
3.1 Linearization and Local Stability

Here we present Lyapunov’s first method, and study the local stability of the system based on this method.

3.1.1 Lyapunov’s First Method

Consider the system described by the equation:

\[ \dot{x} = f(x) \]

\(x\) can be written as:

\[ x = x_e + \delta x \]

Then we have:

\[ \delta \dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x_e} \delta x + h(x_e, \delta x) \]

And if we define: \( A = \left. \frac{\partial f}{\partial x} \right|_{x_e} \) we will have:

\[ \delta \dot{x} = A \delta x + h(x_e, \delta x) \]

Lyapunov proved that the eigenvalues of \( A \) indicate “local” stability of the nonlinear system about the equilibrium point if:

1. \( \lim_{\|\delta x\| \to 0} \frac{\|h(x_e, \delta x)\|}{\|\delta x\|} = 0 \) \( (“The linear terms dominate”) \)
2. There are no eigenvalues with zero real part.

Using the above method, here we consider the stability of the system. First we need to find the equilibrium points of the system. The equilibrium points are obtained by fixing the input, \( u_e \) and then solving for \( x_e \). In this system there are infinite equilibrium points which can be gained by assuming different equilibrium inputs. But we are only interested in one equilibrium point in which the system is working most of the time.

That equilibrium point is obtained by setting the temperature of the room equal to the set point temperature that are assigned by the user (building occupants), and then solving for the equilibrium temperature of the walls and the equilibrium inputs. Here we have ignored the disturbance terms and the equilibrium point is achieved only by setting the nonlinear equation \( f(x_e, u_e) \) equal to zero. We have solved for an equilibrium point near the setpoint temperature \( T_{setpoint} = 22^\circ C \). By solving the mentioned equation we find the equilibrium point to be

\[
X_e = \begin{bmatrix} 22.0666 \\ 17.1058 \\ 17.1058 \\ 17.1058 \\ 17.1058 \end{bmatrix} \quad u_e = 0.00333
\]

Here we need to find the matrix \( A = \frac{\partial f}{\partial x}|_{e} \) and evaluate the above equilibrium point. Matrix \( A \) is found to be:

\[
A = \begin{bmatrix}
\frac{-1}{C_r} \sum_{j=1}^{4} \frac{1}{R_{eij}} - \frac{C_p}{C_r} u_e & \frac{1}{C_r R_{e\alpha1}} & \frac{1}{C_r R_{e\alpha2}} & \frac{1}{C_r R_{e\alpha3}} \\
\frac{1}{C_w R_{w\alpha1}} & \frac{-1}{R_1 C_{w1}} & 0 & 0 & 0 \\
\frac{1}{C_w R_{w\alpha2}} & 0 & \frac{-1}{R_2 C_{w2}} & 0 & 0 \\
\frac{1}{C_w R_{w\alpha3}} & 0 & 0 & \frac{-1}{R_3 C_{w3}} & 0 \\
\frac{1}{C_w R_{w\alpha4}} & 0 & 0 & 0 & \frac{-1}{R_4 C_{w4}}
\end{bmatrix}
\]
We have computed the eigenvalues of this matrix for given values of parameters in MATLAB. The eigenvalues are as follows which indicate the local stability of the system about the desired equilibrium point.

$$\Lambda = \begin{bmatrix} -3.0142 \\ -1.2818 \\ -1.2818 \\ -1.2818 \end{bmatrix}$$

Note that the two conditions of the Lyapunov’s first method are satisfied here, as the linear terms are dominant and there is no eigenvalue with zero real part.
Chapter 4

Controllability and Observability Analysis

4.1 Controllability of the system

There is a theorem in the notes that says, the system defined by:

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]  (4.1)

is locally accessible about \( x_0 \) if the accessibility distribution \( C \) spans \( n \) space, where \( n \) is the rank of \( x \) and \( C \) is defined by:

\[
C = \begin{bmatrix}
g_1 & g_2 & g_3 & \cdots & [g_i, g_j] & \cdots & [ad^k_{g_i}, g_j] & \cdots & [f, g_i] & \cdots & [ad^k_f, g_i] & \cdots
\end{bmatrix}
\]

For the system that we are studying, since we have one input, the accessibility distribution \( C \) is as follows:
We have calculated $g$ in the modeling part. Here we calculate the rest of the accessibility distribution matrix:

\[ [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \]

\[ [f, [f, g]] = \frac{\partial [f, g]}{\partial x} f - \frac{\partial f}{\partial x} [f, g] \]

and

\[ [ad_f^k, g] = [f, [ad_f^{k-1}, g]] \]

Calculation of these matrices are brought in the next pages.
Controllability analysis:
cont. Controllability analysis:
We can see that the controllability analysis for a nonlinear system can be very messy, including many matrix multiplications, which are expensive calculations. In the next section we study the observability of the system.

We could have determined the controllability of the system intuitively without doing these controllability analyses. Since there is no direct input to the other four states $T_{wi} \forall \ i = 1, 2, 3, 4$, therefore the system can not be steered into any arbitrary state around the $x_0$, which means the system is not reachable.

4.2 Observability of the system

In order to determine the observability of the system, according to the theorem in the notes for the single input single output system we have:

$$\dot{x} = f(x, u)z = h(x)$$

we look at the derivatives of $z$:

$$z = h = L_0^0(h)$$
$$\dot{z} = \dot{h} = L_1^1(h)$$
$$\vdots$$
$$z^{(n-1)} = L_f^{n-1}(h)$$

Then we define the $l$ vector as below:

$$l(x) = \begin{bmatrix} L_0^0(h) \\ \vdots \\ L_f^{n-1}(h) \end{bmatrix}$$
Then for local observability about $x_0$ for $u = u_0$, we should have:

\[
dG = O = \frac{\partial l(x, u_0)}{\partial x} \bigg|_{x=x_0}
\]

For the system we are studying the observability analysis is fairly easy, because the output of the system is a linear function of the states, hence $h(x) = Cx$. So we have:

\[
l(x) = \begin{bmatrix}
Cx \\
CAx \\
\vdots \\
CA^{n-1}x
\end{bmatrix}
\]

Therefore the observability matrix $O$ is as follows:

\[
O = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

which needs to have rank $n = 5$ so that the system is observable. The rank of the matrix in computed in MATLAB which shows that the system is not locally observable about $x_0$ which means the nonlinear system is not distinguishable at any state.
Chapter 5

Feedback Linearization

Feedback linearization is an approach to nonlinear control design that has attracted lots of research in recent years. The central idea is to algebraically transform nonlinear system dynamics into (fully or partly) linear ones, so that linear control techniques can be applied. This differs entirely from conventional (Jacobian) linearization, because feedback linearization is achieved by exact state transformation and feedback, rather than by linear approximations of the dynamics [1]. Here we will study both input-output linearization and input-state linearization techniques.

5.1 Input-Output Linearization

Input-output linearization is a control technique where the output $y$ is differentiated until the physical input $u$ appears in the $r^{th}$ derivative of $y$. Then $u$ is chosen to yield a transfer function from the “synthetic input” $v$, to the output $y$ which is:

$$\frac{Y(s)}{V(s)} = \frac{1}{s^r}$$
If the relative degree, is less than \( n \) the order of the system, then there will be internal dynamics. Here we will implement this design algorithm on the system:

\[
y = T_1
\]

\[
\dot{y} = \dot{T}_1
\]

\[
= -\frac{1}{C_r} \left( \sum_{j=1}^{4} \frac{1}{R_{cij}} \right) T_1 + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{cij}} + \frac{c_p}{C_r} (T_0 - T_1) \dot{m}_1 + \frac{1}{C_r} q_{\text{int}}
\]

\[
= v
\]

Since \( u \) appears in the first derivative of the output then the relative degree of the system is 1, and therefore the internal dynamics is of order 4. We can now design a controller for this system, using any linear controller design method. We have

\[
v = \alpha + \beta u
\]

Where \( \alpha, \beta \) and \( u \) are defined as:

\[
\alpha = -\frac{1}{C_r} \left( \sum_{j=1}^{4} \frac{1}{R_{cij}} \right) T_1 + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{cij}} + \frac{1}{C_r} q_{\text{int}}
\]

\[
\beta = +\frac{c_p}{C_r} (T_0 - T_1)
\]

\[
u = \dot{m}_1
\]

Any linear method can be used to design \( v \). Note that the controller that is implemented on the system is obtained by:
\[ u = \frac{1}{\beta(x)}[-\alpha(x) + v] \]

5.1.1 Internal Dynamics

In order to check the internal dynamics we need to consider the closed loop system. The closed loop system equations for any controller design can be obtained by substituting the corresponding value of the synthetic input \( v \) into the system equations. For the P-controller we have:

\[
\begin{align*}
\dot{T}_1 &= v = -K_P(T_1 - T_{1d}) \\
\dot{T}_w^1 &= \frac{T_1}{C_{w1}R_{c1}} - \frac{T_{w1}}{C_{w1}R_1} + \frac{\alpha}{C_{w1}}q_{rad1} + \frac{T_\infty}{C_{w1}R_{co1}} \\
\dot{T}_w^2 &= \frac{T_2}{C_{w2}R_{c2}} - \frac{T_{w2}}{C_{w2}R_2} + \frac{\alpha}{C_{w2}}q_{rad2} + \frac{T_\infty}{C_{w2}R_{co2}} \\
\dot{T}_w^3 &= \frac{T_3}{C_{w3}R_{c3}} - \frac{T_{w3}}{C_{w3}R_3} + \frac{\alpha}{C_{w3}}q_{rad3} + \frac{T_\infty}{C_{w3}R_{co3}} \\
\dot{T}_w^4 &= \frac{T_4}{C_{w4}R_{c4}} - \frac{T_{w4}}{C_{w4}R_4} + \frac{\alpha}{C_{w4}}q_{rad4} + \frac{T_\infty}{C_{w4}R_{co4}}
\end{align*}
\]

We define:

\[ x = \begin{bmatrix} z \\ \xi \end{bmatrix} \]

Where

\[ z = T_1 \quad \xi = \begin{bmatrix} T_{w1} & T_{w2} & T_{w3} & T_{w4} \end{bmatrix}^T \]

Where \( z \in \mathbb{R}^1 \) and \( \xi \in \mathbb{R}^4 \). The normal forms theorem tells us that there exists a \( \xi \) such that:
\[ \dot{\xi} = \psi(z, \xi) \]

Here we look at the zero dynamics which is written as follows:

\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\xi}_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{-1}{R_1C_{w_1}} & 0 & 0 & 0 \\
0 & \frac{-1}{R_2C_{w_2}} & 0 & 0 \\
0 & 0 & \frac{-1}{R_3C_{w_3}} & 0 \\
0 & 0 & 0 & \frac{-1}{R_4C_{w_4}} \\
\end{bmatrix} \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4 \\
\end{bmatrix} + \begin{bmatrix}
\frac{\alpha}{C_{w_1}}q_{rad_1} + \frac{T_\infty}{C_{w_1}R_{co_1}} \\
\frac{\alpha}{C_{w_2}}q_{rad_2} + \frac{T_\infty}{C_{w_2}R_{co_2}} \\
\frac{\alpha}{C_{w_3}}q_{rad_3} + \frac{T_\infty}{C_{w_3}R_{co_3}} \\
\frac{\alpha}{C_{w_4}}q_{rad_4} + \frac{T_\infty}{C_{w_4}R_{co_4}} \\
\end{bmatrix}
\]

The internal dynamics is stable since all the eigenvalues of the above matrix are negative.

### 5.2 Input-State Linearization

In input-state linearization we are looking for an output \( y = z_1(x) \) such that \( y \) has relative degree \( n \). In our system since the only measurable state is the first state, i.e. \( T_1 \), defining a new output would not be practical for the control purposes.
Chapter 6

Sliding Mode Control

The idea of *sliding mode control* is to reduce every system to a first order system that we are trying to force to zero. Then we can use the intuitive control law described above, and furthermore, we can use ANY gain we want, including infinity (perfect tracking, perfect disturbance rejection). The reduction of the order of the system is done through the definition of a new “output” $s$, which looks like a first order system [1]. This new output, $s$, should have the following properties:

1. The relative degree of the output $s$, should be 1, i.e. $u$ should appear explicitly in the expression for $\dot{s}$.

2. $s \to 0$ is the control gain, that is $s$ needs to be designed so that good things happen to the physical output when $s \to 0$.

6.1 Controller Design

Here we are going to design a sliding mode controller so that the output of the system can asymptotically track a desired trajectory. A good choice
for the sliding surface is the following, because \( u \) appears explicitly in \( \dot{s} \) and \( s \to 0 \) is equivalent to \( T_1 \to T_{1d} \).

\[
    s = T_1 - T_{1d}
\]

Taking the derivative of \( s \) yields:

\[
    \dot{s} = \dot{T}_1 - \dot{T}_{1d}
\]

\[
    = \frac{-1}{C_r} \sum_{j=1}^{4} \frac{T_1}{R_{cij}} + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{ij}} + \frac{c_p}{C_r} (T_0 - T_1) \dot{m}_1 + \frac{1}{C_r} q_{int} - \dot{T}_{1d}
\]

We will assume that there is 50\% uncertainty in our knowledge of the internal heat generation, \( q_{int} \), i.e.

\[
    q_{int} = \hat{q}_{int} + \Delta q_{int}
\]

Where

\[
    \left| \frac{q_{int} - \hat{q}_{int}}{\hat{q}_{int}} \right| \leq 0.5 \quad \Rightarrow \quad \left| \Delta q_{int} \right| \leq 0.5 \hat{q}_{int}
\]

For later analyses and for the sake of simplicity, we define

\[
    \gamma = 0.5 \hat{q}_{int} \tag{6.1}
\]

Therefore \( \dot{s} \) can be written as:

\[
    \dot{s} = \frac{-1}{C_r} \sum_{j=1}^{4} \frac{T_1}{R_{cij}} + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{ij}} + \frac{c_p}{C_r} (T_0 - T_1) \dot{m}_1 + \frac{1}{C_r} \hat{q}_{int} + \frac{1}{C_r} \Delta q_{int} - \dot{T}_{1d}
\]
We can find $\dot{m}_1$ from the above equation considering the worst case modeling errors, and we define the sliding condition to be:

$$s\dot{s} \leq -\eta s \operatorname{sgn}(s)$$

Based on this sliding condition and the uncertainty in the knowledge of the internal heat generation we can find the control input as follows:

$$\dot{m}_1 = -\frac{C_r}{(T_0 - T_1)c_p}\left\{-\frac{1}{C_r} \sum_{j=1}^{4} \frac{T_1}{R_{ci}} + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{ij}} + \frac{1}{C_r} \hat{q}_{int} + K \operatorname{sgn}(s) - \dot{T}_1d\right\}$$

where

$$K = \eta + \gamma$$

And $\gamma$ is given in Equation (6.1).

### 6.1.1 Smoothing

In general, chattering must be eliminated for the controller to perform properly. This can be achieved by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface

$$B(t) = \left\{x \mid |s(x, t)| \leq \Phi \right\}$$

where $\Phi$ is the boundary layer thickness. Therefore by replacing $\operatorname{sgn}(s)$ by the term $\operatorname{sat}(\frac{s}{\Phi})$ we guarantee that we stay in a boundary layer, and that I reach the boundary layer in a finite time, and that once I get in the boundary layer I stay there[1].
So the *smooth* control input is given by;

\[ \dot{m}_1 = -\frac{C_r}{(T_0 - T_1)c_p} \left\{ \frac{-1}{C_r} \sum_{j=1}^{4} \frac{T_1}{R_{cij}} + \frac{1}{C_r} \sum_{j=1}^{4} \frac{T_{wj}}{R_{ij}} + \frac{1}{C_r} \dot{q}_{int} + K.\text{sat}\left(\frac{s}{\Phi}\right) - \dot{T}_{1d} \right\} \]

Where K is defined the same as the previous case. In this example it makes more sense to use a smooth control form a practical point of view, because we always want the temperature of the room to be in a reasonable bound, and we can achieve this goal by using smooth sliding mode control.
6.2 Simulation Results and Observations

The simulation of the system was done in Simulink. The system was modeled using simulink blocks as shown in Figure(6.1). Here we will present the results for different values of K and $\Phi$ to see the effect of these two parameters. The desired output is defined to be a sinusoidal function that is shown in next pages.

Figure 6.1: Simulink model of the nonlinear system
1. $K = 0.001$ and $\Phi = 0.5^\circ C$

★ Observation:
In this case $K$ is too small ($K < \eta + \gamma$) and the system output cannot track the desired output. So you can see that $s$ which is the offset from the desired output keeps up increasing.
2. $K = 0.01$ and $\Phi = 0.5^\circ C$

⭐ Observation:
In this case $K$ has been increased but not enough to make the system track the output which means $K$ is still less than $\eta + \gamma$. The deviation of the temperature from the desired trajectory is less than the previous case due to the increase in gain $K$. 

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3. \( K = 0.1 \) and \( \Phi = 0.5^\circ C \)

\[ K = 0.1 \quad \text{and} \quad \Phi = 0.5^\circ C \]

\[ \star \text{Observation:} \]

In this case \( K \) is large enough to track the desired output very closely, but still there is a little deviation from the desired trajectory, although we are in the allowed region for \( s \), defined by \(|s| \leq \Phi\).
4. $K = 1$ and $\Phi = 0.5^\circ C$

★ Observation:
Here $K$ is very large as the system output is completely following the desired trajectory, i.e. $s \approx 0 \ \forall t > 0$. 
5. $K = 0.017$ and $\Phi = 0.5^\circ C$

\[ K = 0.017 \text{ and } \Phi = 0.5^\circ C \]

**Observation:**
In this case we have tuned the gain $K$ such that it has its marginal value. If you increase $K$ you will be drawn more toward the center of the boundary layer, and if you decrease $K$, you will pass the border of the boundary layer. So the marginal value of $K$ is $K_{\text{marginal}} = 0.017$. 
6. $K = 0.001$ and $\Phi = 1^\circ C$

★ **Observation:**
In this experiment we have increase the thickness of the boundary layer. Since $K$ is very small the output cannot track the desired trajectory. You can see that the behavior of the system response is almost the same as the corresponding case with thinner boundary layer.
7. $K = 0.01$ and $\Phi = 1^\circ C$

\[ K=0.01 \text{ and } \Phi=1^\circ C \]

★ Observation:
The same story as the case we had for the corresponding case with $\Phi = 0.5$. 
8. $K = 0.1$ and $\Phi = 1^\circ C$

**Observation:**
K is large enough so that the output of the system tracks the desired output, but still we are not completely tracking it, although we are in the allowed region for $s$. 
9. $K = 1$ and $\Phi = 1^\circ C$

★ Observation:
K is very large for the system, such that the output is completely tracking the desired output.
10. $K = 0.017$ and $\Phi = 1^\circ C$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot.png}
\end{figure}

\textbf{★ Observation:}
We have tuned the gain $K$ and pushed it to its marginal value. You can see that for this case that we have increase the thickness of the boundary layer, still we have: $K_{\text{marginal}} = 0.017$. Comparing this plot with the corresponding one for $\Phi = 0.5$ shows that this marginal value of $K$ is only a function of $\eta$ and $\gamma$, and is independent of $\Phi$, the thickness of boundary layer.
Chapter 7

Conclusion and Future Work

The modeling and nonlinear control design of the heating, ventilation and air conditioning system of a building was studied in this project. The modeling part was done using lumped capacitance method, and the resulting mathematical model was shown to be nonlinear. First of all we investigated the stability of the system about an equilibrium point in which the system is working most of the time. The system was shown to be locally asymptotically stable. In the next chapter we did controllability and observability analyses. The system is not controllable, as the input has direct effect on one of the states and all the five states of the system cannot be steered arbitrarily to any other state. The controllability analysis of the nonlinear system was shown to get very messy and a lot of matrix calculations was involved. So we skipped a thorough controllability analysis and concluded the uncontrollability of the system from its state space realization. We also studied feedback linearization of the system. It was shown that input-state linearization is not a practical choice, so we did an input-state linearization. The new linearized system is of order one, and the internal dynamics has order four. It was shown that the internal dynamics is stable. In the last part, we designed two sliding mode controller for the nonlinear system. One with the signum function in the sliding condition and the second one with smooth saturation function. From a practical point of view, the second controller was a better fit for the system, regarding both implementation and operational aspects. For designing the sliding mode controller we also assumed that we have 50% un-
certainty in the knowledge of heat generation inside the building. In the last part the simulation results are presented which indicate that the marginal gain of the sliding mode controller is $K_{marginal} = 0.017$, meaning that the system is operating on the border of the boundary condition defined by the constant $\Phi$. For gain values greater than this $K_{marginal}$, the parameter $s$ goes more toward the center of the boundary layer, and for gain values less than this $K_{marginal}$ the parameter $s$ passes the border of the boundary layer, i.e. the allowed region defined for $s$.

Future work may include deriving a more detailed mathematical model for the system and consider the effects of uncertainty in our knowledge of the heat flux from outside and study the importance of this modeling uncertainty on the behavior of the system response and the sliding mode controller gain.
Bibliography
