

Appendix: Useful Concepts from Circuit Theory

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1. Introduction

For understanding the physics of sound and the biophysics of acoustic sensors in animals, the concepts of impedance and admittance, impedance matching, transducer, transformer, passive and active, bidirectional coupling, and resonance are widely used. They all arise from the circuit-theory metamodel, which has been applied extensively in acoustics. As the name suggests, the circuit-theory metamodel provides recipes for constructing models. The example applications of the metamodel in this appendix involve a few elementary concepts from classical physics: the adiabatic gas law, Newton's laws of motion and of viscosity, and the definitions of work and of Gibbs free energy. They also involve a few elements of calculus and of the arithmetic of complex numbers. All of these should be familiar to modern biologists and clinicians. For further examples of the application of circuit concepts to acoustical theory and acoustical design, see Baranek (1954), Olsen (1957), or Morse (1981).

1.1 *The Recipe*

Applied to an elementary physical process, such as sound conduction in a uniform medium, the appropriate recipe comprises the following steps:

1. Identify an entity (other than Gibbs free energy, see below) that is taken to be conserved and to move from place to place during the process.
2. Imagine the space in which the process occurs as being divided into such places (i.e., into places in which the conserved entity can accumulate).
3. Construct a map of the space (a circuit graph) by drawing a node (e.g., a small dot) for each place.
4. Designate one place as the reference or ground place for the process.
5. Assign the label "0" to the node representing the ground place.
6. Assign a unique, nonzero real integer (i) to each of the other nodes in the graph.
7. Define the potential difference between each place (i) and the ground place (0) to be F_{i0} , the corresponding Gibbs potential for the conserved entity

(change in Gibbs free energy of the system per unit of conserved entity moved from place 0 to place i).

8. From basic physical principles (or by measurement), determine the relationship between the amount, Q_i , of conserved entity accumulated at place i and the potential F_{i0} .
9. From basic physical principles (or by measurement), determine the relationship between potential difference F_{ij} between each pair of neighboring places and the flow J_{ij} of conserved entity between those places.

The Gibbs free energy of a system is defined as the total work available from the system less that portion of the work that must be done against the atmosphere. For one-dimensional motion from point a to point b , the work (W_{ab}) done by the system is defined by the expression $W_{ab} = \int_{x=a}^b f(x)dx$ where $f(x)$ is the force applied by the system to the object on which the work is being done. The force is taken to be positive if it is applied in the direction in which the values of x increase; otherwise it is taken to be negative. For tables of Gibbs potentials for various conserved entities, see Lewis (1996).

1.2 Example: An Acoustic Circuit Model

As an example application of the recipe, consider the conduction of sound, axially, through an air-filled cylindrical tube with rigid walls. In that case, one might identify the conserved entity as the air molecules that move from place to place along the tube. As air molecules accumulate at a given place, the pressure at that place increases. Assuming that the air pressure outside the tube is uniform and equal to atmospheric pressure, it is convenient to take that place (the outside of the tube) to be the reference place. The SI unit for number of air molecules is 1.0 mol. Therefore, the SI unit for the flow of air molecules will be 1.0 mol/s; and the SI unit for the Gibbs potential will be 1.0 joule/mol. If one takes p_i to be the absolute pressure at place i along the inside of the tube (SI unit 1.0 nt/m² = 1.0 Pa) and p_0 to be the absolute atmospheric pressure, then the sound pressure (Δp_i) at place i will be the difference between those two pressures:

$$\Delta p_i = p_i - p_0 \quad (1)$$

Acousticians usually assume that the processes involved in sound propagation are adiabatic and that sound pressures are exceedingly small in comparison to atmospheric pressure:

$$\frac{\Delta p_i}{p_0} \ll 1.0 \quad (2)$$

The Gibbs potential difference, F_{i0} , between place i and the reference place is given by:

$$F_{i0} = \frac{\Delta p_i}{c_0} \quad (3)$$

where c_0 is the resting concentration of air molecules (SI unit = 1.0 mol/m³).

$$c_0 = \frac{n_0}{V_0} \quad (4)$$

where n_0 is the resting number of air molecules at place i , and V_0 is the volume of place i (assumed to be constant). The resting absolute pressure at place i is taken to be p_0 .

Equation 3 for the Gibbs potential can be derived easily from the equation for work. Invoking the adiabatic gas law, one can derive the relationship between the Gibbs potential at place i and the excess number of air molecules at place i , Δn_i , where

$$\Delta n_i = n_i - n_0 \quad (5)$$

n_i being the total number of air molecules at place i . For $\Delta p/p_0$ exceedingly small, the adiabatic gas law can be stated as follows:

$$\frac{\Delta p_i}{p_0} = \gamma \frac{\Delta n_i}{n_0} \quad (6)$$

where γ is 1.4, the ratio of specific heat of air at constant pressure to that at constant volume. From this,

$$F_{i0} = \frac{\Delta p_i}{c_0} = \frac{\gamma p_0}{c_0 n_0} \Delta n_i \quad (7)$$

In the circuit-theory metamodel, this is a capacitive relationship. The general constitutive relationship for a linear, shunt capacity at place i in any circuit is

$$F_{i0} = \frac{Q_i}{C_i} \quad (8)$$

where the parameter C_i often is called the *capacitance* or *compliance*. In this case, Q_i , the accumulation of conserved entity at place i , is taken to be to be Δn_i ,

$$C_i = \frac{c_0 n_0}{\gamma p_0} \quad (9)$$

Dividing the cylindrical tube axially into a large number of segments of equal volume, V_0 , brings the first eight steps of the metamodel recipe to completion. For the ninth step one might assume that the acoustic frequencies are sufficiently high to preclude significant propagation of drag effects from the wall of the cylinder into the mainstream of axial air flow. In other words, one might ignore the effects of viscosity (see subsection 8.2). Absent viscosity, the axial flow of air molecules would be limited only by inertia. Lumping the total mass of air molecules at a single place, and invoking Newton's second law of motion, one has the following relationship for the flow, J_{ij} , of molecules from place i to its neighboring place, j :

$$\frac{dJ_{ij}}{dt} = \frac{A_0^2 c_0^2}{V_0 \rho_0} F_{ij} \quad (10)$$

where t is time, ρ_0 is the resting density of the air (SI unit equal 1.0 kg/m^3), and A_0 is the luminal area of the cylindrical tube. In the circuit-theory meta-model, this is an inertial relationship. The general constitutive relationship for a linear, series inertia between place i and j in any circuit is

$$F_{ij} = I_{ij} \frac{dJ_{ij}}{dt} \quad (11)$$

where the parameter I_{ij} often is called the *inertia* or *inertance*. In this case,

$$I_{ij} = \frac{V_0 \rho_0}{A_0^2 c_0^2} \quad (12)$$

2. Derivation of an Acoustic Wave Equation from the Circuit Model

Having thus completed the circuit model, one now can extract from it an equation describing sound conduction through the tube. This normally is accomplished by writing explicit expressions for the conservation of (conserved) entity at node i and the conservation of energy in the vicinity of node i , and then incorporating the constitutive relations obtained in steps 8 and 9 of the meta-model recipe. Let node i have two neighbors, node h on its left and node j on its right. Conservation of air molecules requires that

$$J_{hi} - J_{ij} = \frac{dQ_i}{dt} \quad (13)$$

From Equation 8 one notes that

$$\frac{dQ_i}{dt} = C_i \frac{dF_{i0}}{dt} \quad (14)$$

Therefore,

$$J_{hi} - J_{ij} = C_i \frac{dF_{i0}}{dt} \quad (15)$$

Conservation of energy requires that

$$F_{ij} = F_{i0} - F_{j0} \quad (16)$$

Therefore,

$$F_{i0} - F_{j0} = I_{ij} \frac{dJ_{ij}}{dt} \quad (17)$$

2.1 Translation to a Transmission-Line Model

In the manner of Lord Rayleigh, one can simplify further analysis by taking the limit of each side of Equations 15 and 17 as one allows the number of places represented in the model to become infinite and the extent of each place infinitesimal. Each place represented in the circuit model has length L_0 given by

$$L_0 = \frac{V_0}{A_0} \quad (18)$$

Dividing C_i and I_i by L_0 , one has the capacitance and inductance per unit length of the uniform, air-filled tube:

$$\begin{aligned} \hat{C} &= \frac{C_i}{L_0} = \frac{A_0 c_0^2}{\gamma p_0} \\ \hat{I} &= \frac{I_i}{L_0} = \frac{\rho_0}{A_0 c_0^2} \end{aligned} \quad (19)$$

Now, let x be the distance along the tube and let the length of each place be Δx . One can rewrite Equations 15 and 17 as follows:

$$\begin{aligned} -\Delta J &= J_{hi} - J_{ij} = \hat{C} \Delta x \frac{dF}{dt} \\ -\Delta F &= F_{i0} - F_{j0} = \hat{I} \Delta x \frac{dJ}{dt} \end{aligned} \quad (20)$$

Dividing each side of each equation by Δx and taking the limit as Δx approaches zero, one has

$$\begin{aligned} -\frac{d}{dx} J(x,t) &= \lim_{\Delta x \rightarrow 0} \left[-\frac{\Delta J}{\Delta x} \right] = \hat{C} \frac{d}{dt} F(x,t) \\ -\frac{d}{dx} F(x,t) &= \lim_{\Delta x \rightarrow 0} \left[-\frac{\Delta F}{\Delta x} \right] = \hat{I} \frac{d}{dt} J(x,t) \end{aligned} \quad (21)$$

These are the classic transmission-line equations, variations of which have been used widely in neurobiology. Traditionally, they are written in a more general form:

$$\begin{aligned} \frac{dF}{dx} &= -zJ \\ \frac{dJ}{dx} &= -yF \end{aligned} \quad (22)$$

where z is the series impedance per unit length of the transmission line, and y is the shunt admittance per unit length; and F and J are linear transforms of F and J . Although the notions of impedance and admittance are treated here in a general sense, when they involve time derivatives (as they do here), circuit theorists normally apply them to one of two carefully restricted situations: (1)

sinusoidal steady state (a pure tone has been applied for an indefinitely long time); or (2) zero state (a temporal waveform is applied; at the instant before it begins, all F s and J s in the circuit model are equal to zero). For sinusoidal steady state, circuit theorists traditionally use the phasor transform; for zero state they traditionally use the Laplace transform. Yielding results interpretable directly in terms of frequency, the former is widely used in classical acoustics texts.

2.2 Derivation of the Phasor Transform

The definition of the phasor transform is based on the presumption that exponentiation of an imaginary number, such as ix (where i is the square root of -1), obeys the same rules as exponentiation of a real number (such as x). Specifically, it is presumed that e^{ix} can be evaluated by substituting ix for x in the Maclaurin series for e^x . In that case,

$$e^{ix} = \cos x + i \sin x \quad (23)$$

and

$$\cos x = \text{Re}\{e^{ix}\} \quad (24)$$

where $\text{Re}\{z\}$ is the real part of the complex number, z :

$$\begin{aligned} z &= a + ib \\ i &= \sqrt{-1} \\ \text{Re}\{z\} &= a \end{aligned} \quad (25)$$

The phasor transform applies to sinusoidal steady state, with the radial frequency of the sinusoid being ω (SI unit = 1.0 rad/s).

$$\omega = 2\pi f \quad (26)$$

where f is the conventional frequency (unit = 1.0 Hz). One can represent the phasor transformation as follows:

$$\begin{aligned} H(\omega) &= \Phi\{H(t)\} \\ H(t) &= \Phi^{-1}H(\omega) \end{aligned} \quad (27)$$

where the function of frequency, $H(\omega)$, is the phasor transform of the time function $H(t)$. The transform is defined by its inverse

$$H(t) = \Phi^{-1}\{H(\omega)\} \triangleq \text{Re}[H(\omega)e^{i\omega t}] \quad (28)$$

Combining this with Equations 23–25, one finds the following basic transform pairs:

$$\begin{aligned} \Phi\{\cos\omega t\} &= 1 \\ \Phi^{-1}\{1\} &= \cos\omega t \\ \Phi\{A \cos(\omega t + \alpha)\} &= Ae^{i\alpha} = A \cos \alpha + iA \sin \alpha \\ \Phi^{-1}\{Ae^{i\alpha}\} &= \Phi^{-1}\{A \cos \alpha + iA \sin \alpha\} = A \cos(\omega t + \alpha) \end{aligned} \quad (29)$$

Notice that

$$\Phi^{-1}\{Ae^{i\alpha}Be^{i\beta}\} = AB \cos(\omega t + \alpha + \beta) \quad (30)$$

Thus, multiplying the phasor transform of a sinusoidal function of time (of any amplitude and phase) by the factor $Ae^{i\alpha}$ corresponds to two mathematical operations on the function itself: (1) multiplying the amplitude of the function by the factor A , and (2) adding α radians to the phase of the function. Taking the first derivative of any sinusoidal function of time can be described in terms of the same two operations: (1) the amplitude of the function is multiplied by ω , and (2) $\pi/2$ radians (90 degrees) are added to the phase of the function. Thus,

$$\Phi\left\{\frac{dH(t)}{dt}\right\} = \omega e^{i\pi/2}H(\omega) \quad (31)$$

Applying Equation 23, one has

$$e^{i\pi/2} = i$$

$$\Phi\left(\frac{dH(t)}{dt}\right) = i\omega H(\omega) \quad (32)$$

2.3 Phasor Transform of the Transmission-Line Model

Based on Equation 32, one can write the transformed version of Equation 21 as follows:

$$\frac{d}{dx} F(x, \omega) = -i\omega \hat{L}J(x, \omega)$$

$$\frac{d}{dx} J(x, \omega) = -i\omega \hat{C}F(x, \omega) \quad (33)$$

Thus, for the transmission-line model (Eq. 22), under sinusoidal steady state at radial frequency ω , z and y would be written as follows:

$$z = i \omega \hat{L}$$

$$y = i \omega \hat{C} \quad (34)$$

To simplify notation in transformed equations, ω often is omitted in the arguments of the dependent variables J and F . For z and y both independent of x , as they are taken to be in this case, the spatial solutions to these equations usually are written in one of two forms:

Form 1

$$F(x) = F_f(x) + F_r(x) \quad J(x) = J_f(x) - J_r(x)$$

$$J_f(x) = Y_0 F_f(x) \quad J_r(x) = Y_0 F_r(x)$$

$$F_f(x) = Z_0 J_f(x) \quad F_r(x) = Z_0 J_r(x)$$

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{z}{y}}$$

$$F_f(x) = F_f(0)e^{-\sqrt{zy}x} \quad F_r(x) = F_r(0)e^{\sqrt{zy}x} \quad (35)$$

Form 2

$$\begin{aligned}
 F(x) &= F(0) \cosh(\sqrt{zy} x) - Z_0 J(0) \sinh(\sqrt{zy} x) \\
 J(x) &= J(0) \cosh(\sqrt{zy} x) - Y_0 F(0) \sinh(\sqrt{zy} x) \\
 Z_0 &= \frac{1}{Y_0} = \sqrt{\frac{z}{y}}
 \end{aligned} \tag{36}$$

2.4 Forward and Reverse Waves

Define the forward direction for the transmission line as the direction of increasing values of x . In the first form, $F_f(x)$ and $J_f(x)$ are interpreted as being the potential and flow components of a wave traveling in the forward direction; $F_r(x)$ and $J_r(x)$ are interpreted as the components of a wave traveling in the reverse direction. At any instant, the product $F_f(x)J_f(x)$ of the corresponding time functions is the rate at which the forward wave carries Gibbs free energy (in the forward direction) past location x , and the product $F_r(x)J_r(x)$ is the rate at which the reverse wave carries Gibbs free energy past x (in the reverse direction).

2.4.1 Speed of Sound and Characteristic Impedance

For the simple transmission-line model under consideration here, z is purely inertial and y is purely capacitive. The wave components in form 1 of the solution become

$$\begin{aligned}
 F_f(x) &= F_f(0)e^{-i\sqrt{\hat{C}}\omega x} \\
 F_r(x) &= F_r(0)e^{+i\sqrt{\hat{C}}\omega x}
 \end{aligned} \tag{37}$$

Both components are sinusoids, radial frequency ω . For the forward component, the phase at distance x from the origin lags that at the origin by an amount that is directly proportional to x . That is the behavior of a wave traveling at constant speed in the direction of increasing x (the forward direction). For the reverse component, the phase at distance x leads that at the origin by an amount proportional to x . That is the behavior of a wave traveling at constant speed in the direction of decreasing x (the reverse direction). For both components, the amplitude is independent of x . The speed, ϕ , of the wave is the same in both cases. The solution is valid for all frequencies, so one can generalize it to waves of arbitrary shape that travel at constant speed without changing shape:

$$\begin{aligned}
 F_f(x,t) &= F_f\left(0,t - \frac{x}{\phi}\right) & F_r(x,t) &= F_r\left(0,t + \frac{x}{\phi}\right) \\
 J_f(x,t) &= Y_0 F_f(x,t) & J_r(x,t) &= Y_0 F_r(x,t)
 \end{aligned} \tag{38}$$

$$Y_0 = \frac{1}{Z_0} = \sqrt{\frac{\hat{C}}{\hat{I}}} = \frac{A_0 c_0^2}{\sqrt{\rho_0 \gamma p_0}} \tag{39}$$

$$\phi = \frac{1}{\sqrt{\hat{I}\hat{C}}} = \sqrt{\frac{\gamma p_0}{\rho_0}} \quad (40)$$

The parameter Z_0 , usually called the characteristic impedance, is a positive real number in this case. For sinusoidal steady state this implies that at each place (each value of x) along the acoustic path, F_r and J_r are perfectly in phase with one another, as are F_r and J_r . The relationship between Z_0 and the physical parameters of the system and its SI unit both depend on the choice one made at step 1 of the metamodel recipe. If one carries out the model construction correctly, however, the wave speed will be independent of that choice.

3. Alternative Formulations of the Circuit Model

Instead of taking the number of air molecules as the conserved entity, one might have taken their mass. Assuming that the density changes induced in the air by the propagating sound wave are negligible (the analysis in section 2 was based on the same assumption), acousticians often use fluid volume as the conserved entity (as a surrogate for mass). Sometimes mass itself is used. All of these choices lead to equivalent formulations and moving from one to another requires only the application of appropriate multiplicative factors. The SI unit of impedance for each choice of conserved entity is easily determined. If the selected entity were apples, with an SI unit of 1.0 apple, then the SI unit of flow would be 1.0 apple/s, and the SI unit of Gibbs potential would be 1.0 joule/apple. The SI unit of impedance would be that of Gibbs potential divided by flow—1.0 joule s per apple². For the acousticians' choice of volume, the SI unit of conserved entity is 1.0 m³. The SI unit of impedance becomes 1.0 joule s per m⁶. For the choice made at the beginning of section 1.2 (SI unit of conserved entity = 1.0 mol), the SI unit of impedance is 1.0 joule s/mol².

3.1 Conventional Acousticians' Formulation

To obtain the acousticians' J (SI unit 1.0 m³/s), one divides the J of section 2 (SI unit 1.0 mol/s) by c_0 (SI unit 1.0 mol/m³). To obtain the acousticians' F (SI unit 1.0 joule/m³), one multiplies the F of section 2 (SI unit 1.0 joule/mol) by c_0 . When these changes are made, the inertance and compliance per unit length become.

$$\begin{aligned} \hat{C} &= \frac{A_0}{\gamma p_0} \\ \hat{I} &= \frac{\rho_0}{A_0} \end{aligned} \quad (41)$$

and the characteristic impedance, Z_0 , becomes

$$Z_0 = \frac{\sqrt{\gamma p_0 \rho_0}}{A_0} \quad (42)$$

Recognizing γp_0 as the adiabatic bulk modulus of an ideal gas at pressure p_0 , one can translate Equations 40 and 42 into general expressions for compressional (longitudinal) acoustic waves in fluids:

$$\begin{aligned} \hat{C} &= \frac{A_0}{E_B} \\ \phi &= \sqrt{\frac{E_B}{\rho}} \\ Z_0 &= \frac{\sqrt{E_B \rho}}{A_0} \end{aligned} \quad (43)$$

$$E_B \triangleq -V_0 \frac{\Delta p}{\Delta V} \quad (44)$$

where ϕ is the speed of sound in the fluid; Z_0 is the characteristic acoustic impedance of the fluid; E_B is the fluid's adiabatic bulk modulus; ρ is its density, and A_0 is the cross-sectional area of the acoustic path (the luminal area of the cylindrical tube in the system being modeled here). Once again, these expressions were based on the presumption that changes in the fluid density, ρ , produced by the waves are exceedingly small.

3.2 Modeling Perspective

Following a path already well established by physicists, with the circuit-theory metamodel as a guide, the development to this point has combined that presumption (exceedingly small changes in fluid density) with two conservation principles (energy is conserved, matter is conserved) and with two of the empirical linear laws of physics (the adiabatic form of the ideal gas law—translated to a form of Hooke's law, and Newton's second law of motion) to reconstruct a theory of sound propagation. With the theory constructed, mathematics was used to determine where that combination of presumptions, principles, and empirical laws would lead. What the analysis showed was that the local elastic and inertial properties of fluids (gases and liquids) should combine to allow spatially extensive sound propagation. It also showed that if that were the case, then the speed of propagation should be related to the elastic and inertial parameters (adiabatic bulk modulus and density) in the specific manner described in Equation 43. In fact, the derived relationship (Eq. 43) predicts extremely well the actual speeds of sound in fluids (gases and liquids). This should bolster one's faith not only in the specific theory, but also in the modeling process itself. It follows the pattern established by Isaac Newton and practiced successfully by celebrated physical scientists over and over again since his time. Newton com-

combined a presumption about gravity with a conservation law (momentum is conserved) and with his second law of motion to construct a theory of planetary motion. He used mathematics to see where that theory would lead. Where it led was directly to Kepler's laws, which are empirical descriptions of the actual motions of planets. Recently, writing about Newton's *Principia*, where this theory and its analysis first were published, a noted physicist said that the astounding conclusion one would draw from the work is the fact that nature obeys the laws of mathematics. I would put it differently. Physical nature appears to behave in an astoundingly consistent manner. In mathematics, mankind has invented tools that allow concise and precise statements of apparently consistent local behaviors (e.g., empirically derived laws), tools (such as the circuit-theory metamodel) that allow those statements to be combined to construct theories of spatially (in its general sense) more extensive behaviors, and tools to derive from those theories their predictions. The fact that the predicted behaviors so often match, extremely well, observed behaviors, is attributable to physical nature's consistency.

Returning to the metamodel and to the first step in the recipe, one can see that momentum (Newton's choice) is a particularly interesting alternative to be taken as the conserved entity. Newton's second law equates the force two particles exert on one another to the flow of momentum between them, and (for nonrelativistic systems) the Gibbs potential difference for that momentum flow is the difference between the particles' velocities. When one chooses fluid mass, fluid volume, or number of fluid molecules as the conserved entity, J is proportional to velocity and F is proportional to force. With momentum as the choice, F is proportional to velocity, J is proportional to force. This is an example of the celebrated duality that arises in the circuit-theory metamodel.

4. Generalization to Plane Waves

As long as the consequences of viscosity were negligible, the presence of the rigid cylindrical walls posited in the previous subsections had only one effect on the propagation of sound waves through the fluid inside the cylinder. It forced that propagation to be axial. The transmission-line equations derived for the fluid in the cylinder would apply perfectly well to propagation of a plane wave through any cross-sectional area A_0 , normal to the direction of propagation, in an unbounded fluid medium (e.g., air or water). The propagation speed derived for the waves confined to the fluid in the cylinder is the same as it would have been for unconfined plane waves in the same fluid. The expression for characteristic impedance (Eq. 44) would apply to an interface, normal to the direction of propagation, with cross-sectional area A_0 . Viewed locally, propagating sound waves usually approximate plane waves, making Equation 44 and its antecedents very general. In an unbounded fluid, however, the propagation of sound waves is not confined to a single coordinate. A local volume in such a fluid could have sound waves impinging upon it from all directions at once.

The instantaneous local sound-pressure component of each of these waves is a scalar, and the pressures from all of the waves together would sum to a single value. The local volume flow component of each wave will now be a vector (having an amplitude and a direction), and the flows from all of the waves would sum vectorially, at any given instant (t), to a single amplitude and direction. For a sound wave from a given direction, the local flow of Gibbs free energy through a cross-sectional area A_0 would be given by the product of the local flow vector ($\mathbf{J}(t)$) for that wave and the corresponding local Gibbs potential ($F(t)$) for that wave. Notice that the product itself now is a vector, with amplitude and direction. The direction of the vector is the direction of energy flow. Notice that it is the same as the direction of $\mathbf{J}(t)$ whenever $F(t)$ is positive. The cross-sectional area, A_0 , would be taken in a plane normal to the direction of energy flow.

4.1 Particle Velocity

Noting that $\mathbf{J}(t)$ for a uniform plane wave would be directly proportional to A_0 , one can normalize it to form another variable—flow density ($\mathfrak{D}(t)$).

$$\mathfrak{D}(t) = \frac{\mathbf{J}(t)}{A_0} \quad (45)$$

When \mathbf{J} is volume flow (the acousticians' choice, SI unit = 1.0 m³/s), \mathfrak{D} is the sound-induced component of the velocity of the fluid (SI unit = 1.0 m/s). In that case, it traditionally is labeled the *particle velocity*. F , in that case (volume taken to be the conserved entity), is the local sound pressure. Whatever the choice of conserved entity, the instantaneous product $F\mathfrak{D}$ will be a vector whose amplitude is the instantaneous flow of Gibbs free energy per unit area (SI unit = 1.0 W/m²), and whose direction is the direction of that energy flow.

It is interesting to contemplate the vector sum of particle velocities of several unrelated sound waves impinging on a small local volume from different directions in an extended fluid. The local sound pressure would vary in time much as it might for a plane wave. The particle-velocity vector, on the other hand, would vary not only in amplitude, but also in direction. Try to imagine the computational task one would face trying to reconstruct the original sound waves from the combination of that scalar function, $F(t)$, and that vector function, $\mathfrak{D}(t)$.

4.2 Particle Displacements

For an acoustical sine wave, the local particle velocity is a sinusoidal function of time, as is the corresponding local particle displacement, $\delta(x,t)$. One can compute the amplitude, $D(x)$, of the latter directly from the amplitude, $B(x)$, of the former:

$$\begin{aligned} \vartheta(x,t) &= B(x)\cos(\omega t) \\ \delta(x,t) &= D(x)\sin(\omega t) = \int \vartheta(x,t)dt = \frac{B(x)\sin(\omega t)}{\omega} \\ D(x) &= \frac{B(x)}{\omega} \end{aligned} \quad (46)$$

This, in turn, can be related to the local peak sound pressure, $\Delta p(x)$, through the characteristic impedance:

$$\begin{aligned} F(x,t) &= \Delta p(x)\cos(\omega t) \\ \vartheta(x,t) &= \frac{J(x,t)}{A_0} = \frac{F(x,t)}{A_0 Z_0} = \frac{F(x,t)}{\sqrt{E_B \rho}} \\ B(x) &= \frac{\Delta p(x)}{\sqrt{E_B \rho}} \\ D(x) &= \frac{\Delta p(x)}{\omega \sqrt{E_B \rho}} \\ \sqrt{E_B \rho} &= \rho \phi = \frac{E_B}{\phi} \end{aligned} \quad (47)$$

In water, ρ is approximately 10^3 kg/m^3 and ϕ is approximately $1.5 \times 10^3 \text{ m/s}$, making $\rho\phi$ approximately $1.5 \times 10^6 \text{ kg/m}^2\text{s}$. Standard pressure of 1.0 atm is defined to be 101,325 Pa. The velocity of sound in dry air at 1.0 atm and 20°C is approximately 340 m/s. Therefore, for dry air at 1.0 atm and 20°C , E_B/ϕ (which is $1.4\rho\phi$) is approximately $400 \text{ kg/m}^2\text{s}$. For a given sound pressure, this means that the peak local particle displacement in air will be almost 4000 times as great as it is in water.

For a 100-Hz tone with a peak sound pressure of 1.0 Pa (the upper end of the usual sound intensity range), for example, the peak local particle displacement in water would be approximately 10 nm. In air, it would be 40 μm . For 1-kHz tones, those particle displacements would be 1 nm and 4 μm , respectively, and for 10 kHz they would be 0.1 nm and 0.4 μm . At the nominal threshold level for human hearing (0.00003 Pa) at 1.0 kHz, they would be 0.03 pm and 0.12 nm. If an evolving animal were able to couple a strain-sensitive device (neural transducer) directly to a flexible structure (e.g., a cuticular sensillum, an antenna, hair) projecting into the air, then a sense of hearing could evolve based on direct coupling of particle-velocity to that structure. One would guess that such a sense would be limited to low-frequency, high-intensity sounds. For a strain-sensitive device (e.g., a bundle of stereovilli) projected into water, on the other hand, an evolving sense of hearing based on direct coupling of particle velocity to that device would be much more limited (4,000 times less effective at every frequency, every intensity). For such a device, one might expect evolution to have incorporated auxiliary structures to increase the displacement of the strain-sensor well beyond the particle displacement of the acoustic wave in the medium. For detection of airborne sound, such auxiliary structures also would have to include a mechanism for transferring acoustic energy from the

air to the strain sensor. In the language of the circuit-theory metamodel, the latter would be a *transducer*, the former a *transformer* or an *amplifier*.

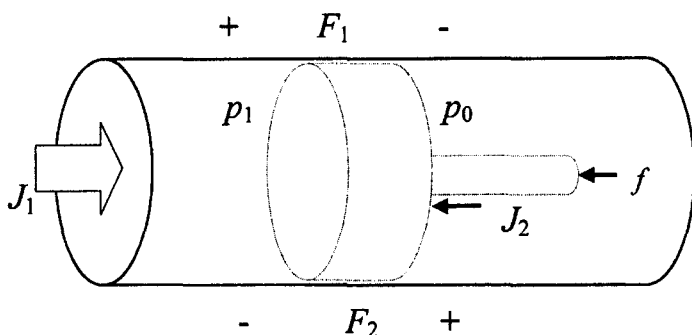
5. Passive Transducers

In the basic circuit model, one tracks the flow of a single entity from place to place. By taking F to be the Gibbs potential for that entity, one is able to track the flow of Gibbs free energy as well. The conserved entity will be confined to a particular realm. Air molecules, for example, are confined to the volume containing the air, which one might label the *pneumatic realm*. The Gibbs free energy (acoustic energy) that they carry, on the other hand, can be transferred in and out of that realm. In the circuit-theory metamodel, a device that transfers energy from one physical realm to another is a transducer. A piston can be modeled as such a device. It can translate sound pressure, F_1 , and flow of air molecules, J_1 , in a pneumatic realm to a Gibbs potential, F_2 , and flow, J_2 , appropriate to a translational rigid-body mechanical realm, and vice versa. In doing so, it carries Gibbs free energy back and forth between those two realms. For a rigid-body mechanical realm, there are two common choices for conserved entity: (1) the shapes of the rigid elements, or (2) momentum. In rigid-body mechanics, it is convenient to treat each of the orthogonal directions of translational and rotational motion as a separate realm. For the piston, the motion of interest is axial (translational motion perpendicular to the face of the piston). Following standard engineering practice, one would select axial displacement as the conserved entity (as a surrogate for shape, displacement is conserved as it passes through an element whose shape does not change). The SI unit of displacement is 1.0 m. The flow, J_2 , becomes axial velocity (SI unit = 1.0 m/s); and the Gibbs potential, F_2 , becomes force (SI unit = 1.0 joule/m). The directions assigned in Fig. A.1 are the conventional associated reference directions for the circuit-theory construction known as a two-port element. When F_1 is positive, it tends to push the piston to the right, the direction assigned to positive values of J_1 . When F_2 is positive, it tends to push the piston to the left, the direction assigned to positive values of J_2 .

Constitutive relationships for the piston can be expressed conveniently in the following form:

$$\begin{aligned} F_1 &= h_{11}J_1 + h_{12}F_2 \\ J_2 &= h_{21}J_1 + h_{22}F_2 \end{aligned} \quad (48)$$

where the parameters h_{12} and h_{21} are transfer relationships; h_{11} is an impedance (ratio of a Gibbs potential to the corresponding flow); and h_{22} is an admittance (ratio of a flow to the corresponding Gibbs potential). One estimates these parameters individually (by measurement or by invoking simple physics) from the following equations:



$$F_1 = p_1 - p_0$$

$$F_2 = f$$

FIGURE A.1. Depiction of a piston inside a cylinder. From the acoustic (fluid) realm, the piston is subjected to a pressure difference, $p_1 - p_0$; and translational motion of the piston corresponds to volume flow, J_1 , of the fluid. From the rigid-body mechanical realm, the piston is subjected to a force, f , and translational motion of the piston corresponds to translational velocity, J_2 .

$$\begin{aligned}
 h_{11} &= \left. \frac{F_1}{J_1} \right|_{F_2=0} & h_{12} &= \left. \frac{F_1}{F_2} \right|_{J_1=0} \\
 h_{21} &= \left. \frac{J_2}{J_1} \right|_{F_2=0} & h_{22} &= \left. \frac{J_2}{F_2} \right|_{J_1=0}
 \end{aligned} \tag{49}$$

If, for now, one assumes that the piston is perfectly rigid, that its mass is negligible, and that it operates with no resistance, then h_{11} and h_{22} can be taken to be zero. That leaves one with the equations for an ideal transducer:

$$F_1 = h_{12}F_2 \quad J_2 = h_{21}J_1 \tag{50}$$

5.1 Reciprocity

Onsager (1931) demonstrated that, ideal or not, as long as it does not involve spinning elements (e.g., a gyroscope or spinning charge), a linear, passive transduction process such as this must be reciprocal. For the selected parameters (applied to the associated reference directions of Fig. A.1), that means

$$h_{12} = -h_{21} \tag{51}$$

As a specific test of the Onsager reciprocity theorem, one can invoke the acousticians' choice of conserved entity (volume) in the pneumatic realm, so that F_1 is the sound pressure on the left side of the piston ($F_1 = \Delta p_1 = p_1 - p_0$) and J_1 is the volume flow on the left. From elementary physics one deduces that J_1

will be zero when F_1 (the net pressure pushing from the left) times the luminal area, A_0 , is equal to the force pushing from the right. Therefore,

$$h_{12} = \left. \frac{F_1}{F_2} \right|_{J_1=0} = \frac{1}{A_0} \quad (52)$$

When there is no force from the right (F_2 is zero), so that the piston could not be deformed even if it were not perfectly rigid, then from elementary geometry one knows that the (rightward) volume flow into the piston, J_1 , must equal the luminal area, A_0 , times the rightward velocity of the piston. Therefore,

$$h_{21} = \left. \frac{J_2}{J_1} \right|_{F_2=0} = -\frac{1}{A_0} \quad (53)$$

and $h_{12} = -h_{21}$, as the Onsager theorem says it must.

5.2 Bidirectionality and Its Implications: Driving-Point Impedance

Reciprocity implies bidirectionality (Gibbs free energy can be transferred in both directions across the transducer), which in turn implies that what happens in either of the physical realms connected to a transducer will affect what happens in the other physical realm. This implication becomes explicit in the expressions for driving-point impedance or admittance. Imagine an acoustic source on the left side of the piston, and mechanical impedance (e.g., a combination of masses, springs, and dashpots) on the right, as in Fig. A.2. Here, the mechanical impedance is designated simply Z_M . For the general linear situation, one would have

$$\begin{aligned} F_1 &= h_{11}J_1 + h_{12}F_2 \\ J_2 &= h_{21}J_1 + h_{22}F_2 = -\frac{F_2}{Z_M} \end{aligned} \quad (54)$$

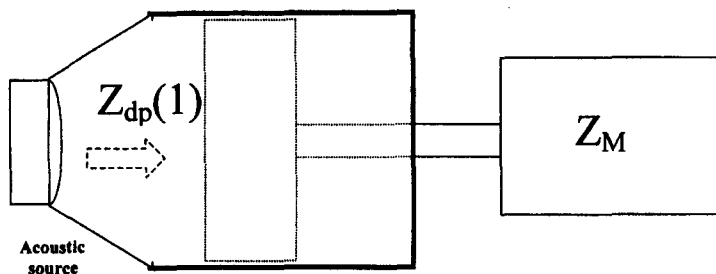


FIGURE A.2. Depiction of an acoustic source connected to a mechanical load (impedance Z_M) through a piston. The resulting pneumatic impedance, $Z_{dp}(1)$, faced by the source depends on both Z_M and the properties of the piston (see text).

from which

$$Z_{dp}(1) = \frac{F_1}{J_1} = h_{11} - \frac{h_{12}h_{21}}{h_{22} + 1/Z_M} \quad (55)$$

where $Z_{dp}(1)$ is the (driving-point) impedance that the acoustic source faces as it drives the transducer from the pneumatic realm. It should not be surprising that $Z_{dp}(1)$ depends strongly on what is connected to the transducer in the other realm (i.e., on Z_M). For this passive transducer (which has no spinning elements), with h_{11} and h_{22} both equal zero,

$$Z_{dp}(1) = \frac{Z_M}{A_0^2} \quad (56)$$

Note that with the current choices of conserved entities (volume in the pneumatic realm, axial displacement in the rigid-body mechanical realm) the SI unit of $Z_{dp}(1)$ is 1.0 joule s per m⁶ and that of Z_M is 1.0 joule s per m². The factor $1/A_0^2$ translates the rigid-body mechanical impedance, Z_M , into the pneumatic impedance, $Z_{dp}(1)$.

6. Terminal Impedances and Reflected Waves

Imagine the piston with its very short cylinder (luminal area A_0) connected to the end ($x = L$) of an air-filled tube (also with luminal area A_0). $Z_{dp}(1)$ now forms what is known as a *terminal impedance* for the acoustic path through the air in the tube. Applying form 1 of the transmission-line equation (Eq. 35), one has

$$\begin{aligned} F_1 &= F_r(L) + F_i(L) = Z_{dp}(1)J_1 & J_1 &= J_r(L) - J_i(L) \\ F_r(L) &= Z_0 J_r(L) & F_i(L) &= Z_0 J_i(L) \end{aligned} \quad (57)$$

where F_1 and J_1 are the sound pressure and volume flow applied to the piston by the air in the tube; Z_0 is the characteristic impedance of the acoustic path through the air in the tube. From Equation 57 one can derive the reflection coefficient $F_r(L)/F_i(L)$ of the air-transducer interface:

$$\frac{F_r}{F_i} = \frac{Z_{dp}(1) - Z_0}{Z_{dp}(1) + Z_0} \quad (58)$$

If the goal is to maximize the transfer of Gibbs free energy from the pneumatic realm (acoustic energy) to the rigid-body mechanical realm (mechanical energy), then one wants to minimize the amplitude of the reflected wave, F_r . It is clear from Equation 58 that F_r will be reduced to zero if $Z_{dp}(1) = Z_0$. In other words, there will be no reflection if the acoustic path is terminated with an impedance equal to the characteristic impedance of the path. If that is not so, one often can make it so by inserting an appropriate passive transformer on one side or the other of the transducer (the piston).

6.1 Transformers and Impedance Matching

The simplest passive transformers are constructed as a cascade of two passive transducers connected back to back. Consider the system depicted in Figure A.3. Here are two rigid-walled cylindrical tubes containing different fluids. Between the two fluids is a cascade of two pistons connected back to back in a translational rigid-body mechanical realm. Being reversed, the second piston is represented by a two-port model in which F_1 is force, J_1 is rightward translational (axial) velocity, F_2 is pressure, and J_2 is leftward volume flow. In that case, if one assumes as before that the piston is perfectly rigid, that its mass is negligible, and that it operates with no resistance,

$$h_{11} = 0 \quad h_{12} = A_2 \quad h_{21} = -A_2 \quad h_{22} = 0 \quad (59)$$

Applying Equation 55 to this situation (Z_M becomes Z_{02}), one finds that the mechanical driving-point impedance at the left side of the second piston is

$$Z_{dp}(p2) = A_2^2 Z_{02} \quad (60)$$

Applying the same equation to the first piston (the piston is not reversed, Z_M becomes $Z_{dp}(p2)$), one finds that the driving-point impedance at the left side of the complete transformer is

$$Z_{dp}(p1) = \frac{A_2^2}{A_1^2} Z_{02} = \tau^2 Z_{02} \quad (61)$$

To avoid reflections of sound waves as they pass from fluid 1 to fluid 2 through the transformer formed by the pistons, one requires

$$\frac{A_2^2}{A_1^2} Z_{02} = Z_{01} \quad (62)$$

Substituting the expression for Z_0 from Equation 44, one can restate this requirement in terms of the densities and adiabatic bulk moduli of the two fluids:

$$A_2 \sqrt{\rho_2 E_{B2}} = A_1 \sqrt{\rho_1 E_{B1}} \quad (63)$$

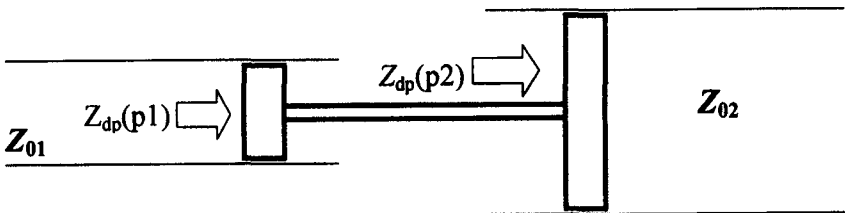


FIGURE A.3. Depiction of an acoustic transformer comprising back-to-back pistons rigidly connected in the rigid-body mechanical realm. The transformer couples an acoustic medium with characteristic impedance Z_{01} to one with characteristic impedance Z_{02} .

The symmetry of this relationship implies that when it is true, sound waves pass through the transformer in either direction without reflection.

Equation 61 is a form of the classic transformer equation, τ being the *transformer ratio*. By adjusting τ (the ratio of piston areas), one could, in principle, transform any real-valued impedance (Z_{02}) to match any other real-valued impedance (e.g., Z_{01}). A mechanical transformer in which τ is less than 1.0 is said to provide *mechanical advantage*. The passive device that does so comprises either a single pair of back-to-back passive transducers (a simple machine), or multiple pairs of such transducers (a compound machine). In addition to pistons and their relatives (including flexible diaphragms), the repertoire of passive mechanical transducers includes the wheel (transfers free energy between translational and rotational realms), which is the basis of gear systems and pulley systems, and the semilever—a rigid bar between a fulcrum and a point of applied force (also transfers free energy between translational and rotational realms), which is the basis of lever systems. For gear or pulley systems, τ is the ratio of wheel (e.g., pulley or gear) radii; for lever systems it is the ratio of semilever lengths.

7. Driving-Point Impedance of a Terminated Acoustic Path

In the previous subsection, the piston was modeled as a circuit element with two distinct ports: port 1 facing the pneumatic realm, and port 2 facing the rigid-body mechanical realm. The same sort of model can be useful as well in a single realm. Suppose the length of a fluid-filled, rigid-walled cylindrical tube were L . If one were concerned only with the acoustic variables (F s and J s) at the ends of the tube, one could translate the transmission-line model (Eq. 36) to a two-port model, one port representing one end of the tube, the other port representing the other end. For that purpose, it is useful to use the two-port impedance (z) parameters instead of the two-port hybrid (h) parameters:

$$F_1 = z_{11}J_1 + z_{12}J_2 \quad F_2 = z_{21}J_1 + z_{22}J_2 \quad (64)$$

where F_1 and J_1 in Equation 64 correspond to $F(0)$ and $J(0)$, respectively, in Equation 36; F_2 and J_2 in Equation 64 correspond to $F(x)$ and $J(x)$ in Equation 36 when x is set equal to L . The corresponding z parameters are derived as follows:

$$z_{11} = \left. \frac{F_1}{J_1} \right|_{J_2=0} = \left. \frac{F(0)}{J(0)} \right|_{J(L)=0}$$

$$J_2 = J(L) = 0 = J(0)\cosh(\sqrt{zy} L) - Y_0 F(0)\sinh(\sqrt{zy} L)$$

$$z_{11} = Z_0 \frac{\cosh(\sqrt{zy} L)}{\sinh(\sqrt{zy} L)}$$

$$z_{21} = \left. \frac{F_1}{J_2} \right|_{J_1=0} = \left. \frac{F(0)}{J(L)} \right|_{J(0)=0} = \frac{Z_0}{\sinh(\sqrt{zy} L)} \quad (65)$$

and, by symmetry, $z_{22} = z_{11}$, $z_{21} = z_{12}$. The equality of z_{12} and z_{21} (or, equivalently, $h_{12} = -h_{21}$) also is stated in a general reciprocity theorem that applies to all passive, linear circuit models in which rotation (sometimes called gyration) is not represented. Proof of the theorem can be found in standard circuit-theory texts. For the z parameter set, Equation 55 becomes

$$Z_{dp}(1) = \frac{F_1}{J_1} = z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L} \quad (66)$$

where Z_L is the (*load*) impedance connected to port 2. In this case, Z_L is the external impedance that the sound wave encounters as it reaches the far end of the tube (where $x = L$). Combining Equations 65 and 66, one finds the general expression for the driving-point impedance that an acoustic wave would face as it enters the tube (at $x = 0$).

$$Z_{dp} = Z_0 \frac{Z_L \cosh(\sqrt{zy} L) + Z_0 \sinh(\sqrt{zy} L)}{Z_0 \cosh(\sqrt{zy} L) + Z_L \sinh(\sqrt{zy} L)} \quad (67)$$

Notice that if Z_L equals Z_0 , so that there is no reflected wave, then Z_{dp} also equals Z_0 . What Equation 67 describes is the impact of the reflected wave on the driving point impedance when Z_L does not equal Z_0 . For the fluid-filled tube, with a steady tone (frequency $\omega_0 = 2\pi f_0$) applied to the near end ($x = 0$), the driving-point impedance becomes

$$Z_{dp} = Z_0 \frac{Z_L \cos(2\pi \frac{L}{\lambda}) + iZ_0 \sin(2\pi \frac{L}{\lambda})}{Z_0 \cos(2\pi \frac{L}{\lambda}) + iZ_L \sin(2\pi \frac{L}{\lambda})} \quad (68)$$

where λ is the wavelength of the tone in the fluid (e.g., in air or in water):

$$\lambda = \frac{\phi}{f_0} = \frac{2\pi\phi}{\omega_0} \quad (69)$$

As the frequency of the tone increases and L/λ shifts from 0 to 0.25, Z_{dp} will shift from Z_L to Z_0^2/Z_L . For intermediate values of L/λ , the values of Z_{dp} will be complex numbers, implying phase differences between F_1 and J_1 . The size of the phase difference will depend strongly on the frequency of the tone.

7.1 The Significance of Phase Difference: Passivity Constraint

To understand the significance of a phase difference between F_1 and J_1 , one can consider the flow, $P_1(t)$, of energy into the tube at port 1, instant by instant:

$$P_1(t) = F_1(t)J_1(t) \quad (70)$$

From this, one computes the total energy, E_1 , entering the tube over each full cycle of the sinusoidal waveform. Let

$$\begin{aligned} J_1(t) &= A \cos \omega t \\ Z_{dp} &= K e^{i\beta} \\ F_1(t) &= KA \cos(\omega t + \beta) \end{aligned} \quad (71)$$

In that case,

$$E_1 = \int_0^{1/f_0} KA^2 \cos(2\pi f_0 t) \cos(2\pi f_0 t + \beta) dt = \frac{A^2}{2f_0} K \cos \beta = \frac{A^2}{2f_0} \operatorname{Re}\{Z_{dp}\} \quad (72)$$

E_1 is maximum when $\beta = 0$; it is positive when $0 < \beta < \pi/2$ or $0 > \beta > -\pi/2$; it is zero when $\beta = \pi/2$ or $\beta = -\pi/2$; and it is negative when $\pi/2 < \beta < \pi$ or $-\pi/2 > \beta > -\pi$. The combination of the tube and Z_L , as posited here, is passive; it cannot deliver more acoustic energy output than that which has been put into it. Therefore,

$$E_1 \geq 0$$

Passivity constraint

$$-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \quad (73)$$

Which requires simply that

$$\operatorname{Re}\{Z_{dp}\} \geq 0 \quad (74)$$

If Z_{dp} were purely inertial or purely capacitive, then $\operatorname{Re}\{Z_{dp}\}$ would be zero. In that case, the energy delivered into the tube during one half cycle will be delivered back out again during the other half cycle.

7.2 The Evolving Ear's Options for Maximizing Free-Energy Transfer

If, under sinusoidal steady-state conditions, one wishes to transfer Gibbs free energy from one acoustic path to another, and if the second path is finite in length (terminated), and if one wishes to use a transformer to translate the driving-point impedance of the second path to the characteristic impedance of the first, so as to maximize free-energy transfer over a wide range of frequencies, then Equations 68 and 72 establish the constraints that are imposed by the physics of the situation. The driving-point impedance of the second path must be a positive real number that is independent of frequency. Assuming Z_0 is such a number, there seem to be two ways to make that happen: (1) terminate the second path with an impedance equal to the characteristic impedance of that path, or (2) make Z_L real and make the second path so short that L/λ remains very small for all frequencies of interest. These would have been the possibil-

ities open to evolution as it sculpted the paths and transformers for conveying acoustic energy into the ear.

8. Wall Effects

8.1 *Effects of Wall Compliance*

If the wall of the cylinder in the original scheme were compliant, if the diameter of the cylinder were small in comparison to the wavelengths of the sound, and if the pressure outside the wall remained fixed at p_0 (e.g., atmospheric pressure), then the expression for capacitance per unit length (Eq. 43) would become

$$\hat{C} = \frac{A_0}{E_B} + \hat{C}_{\text{wall}} \quad (75)$$

In other words, the compliance per unit length of the wall would add to that of the fluid. In the original formulation, the mass-related conserved entity (mass, particle-population, or volume) was accumulated in local compression of the fluid alone. Now it also is accumulated in local expansion of the cylinder wall. Thus, for a given incremental change in pressure, more of the conserved entity would be stored per unit length. This would make both the speed of the sound and the characteristic impedance of the sound path smaller than they would be in an extended volume of the fluid (see Eqs. 39 and 40). In the mammalian cochlea, the local characteristic impedance and speed of the sound wave evidently are determined almost entirely by the local compliance of the wall (the basilar membrane in that case), the contributions of the endolymph and perilymph compliance (i.e., the compliance of water) being negligible in comparison to that of the wall. Recall that the wavelength of a tone equals the speed of sound divided by the frequency of the tone. If the local speed of sound in the cochlea were that of the inner-ear fluids (i.e., in the neighborhood of 1500 m/s), then the wavelength of a 10-kHz tone, for example, would be in the neighborhood of 15 cm—several times the length of the cochlea. In modern cochlear models, the actual wavelength of a 10-kHz tone is a small fraction of the total cochlear length, implying that

$$C_{\text{wall}} \gg \frac{A_0}{E_B} \quad (76)$$

and that the bulk modulus of water contributes almost nothing to the characteristic impedance or the speed of the wave. Among other things, this means that the local particle displacements for waves propagating in the cochlear fluids will be much greater than for similar waves propagating in unbounded aqueous media (see subsection 4.2).

8.2 *Effects of Viscosity*

For a sinusoidal wave propagating axially in the forward direction through the fluid in the cylinder, the local volume velocity vector, $J(x,t)$, would reverse

periodically (with the same period as the sine wave). The flow would be in the forward direction for one half cycle (when the local sound pressure, F , was positive), and in the opposite direction for the next half cycle (when the local sound pressure was negative). In developing the transmission-line model (see Eqs. 10 and 17) one assumed that the frequencies of the sign waves were sufficiently high to avoid the effects of viscosity. Under this assumption, the local axial particle velocity, $\vartheta(x,t)$, is independent of distance from the wall of the cylinder. Strictly speaking, this implies the existence of an infinite radial gradient of velocity at the interface between the wall and the fluid. Fluid dynamicists generally assume that the particle velocity of the fluid immediately adjacent to the wall must be zero (i.e., that the fluid sticks to the wall), which would move the infinite gradient slightly into the fluid itself. Newton's law of viscosity states that shearing tension, $\tau_{\theta z}$ (tangential force per unit area), between fluid layers is directly proportional to the particle velocity gradient taken normal to those layers. For the fluid in the cylinder, this translates to

$$\tau_{\theta z} = \eta \frac{d\vartheta}{dr} \quad (77)$$

r is the where radial coordinate, with its origin taken to be the central axis of the cylinder, and η is the viscosity of the fluid. What one has assumed, then, are extremely large (near-infinite) shearing forces in the fluid layers immediately adjacent to the wall. These would impose a drag on the more central layers, slowing them down and thus reducing the shear gradient, $d\vartheta/dr$. In this manner, the drag effects would move radially inward, toward the central axis of the cylinder. If the duration of one half cycle of the propagating sine wave were sufficiently short, the drag effects would not have time to spread far from the wall. At the end of the half cycle, the particle velocity would reverse and the spreading process would start over again, with shearing forces in the opposite direction. Throughout the full period of the sine wave, the local axial particle velocity would be nearly independent of r over most of the cross-sectional area (A_0) of the cylinder. In the development of Equations 10 to 12, it had been assumed to be independent of r over the entire area.

If the period of the propagating sine wave were sufficiently long, on the other hand, the drag effects would have time to reach the central axis of the cylinder and approach very close to a steady-state condition known as fully developed flow. Because its effects take time to propagate to the central axis, however, the viscous drag on the wall does not become fully effective until the fluid has moved a sufficient distance from the entrance of the cylindrical tube. For small-amplitude acoustic waves, that distance is comparable to or less than the diameter of the tube (e.g., see Talbot 1996). When the flow through the cylinder is fully developed, the particle velocity is a parabolic function of r (the radial velocity profile has the shape of a parabola centered on the central axis of the cylinder). The relationship between F (pressure) and J (axial volume flow) in that case will be given by Poiseuille's equation for flow in a cylinder,

$$\frac{dF}{dx} = -z_R J$$

$$z_R = \frac{8\pi\eta}{A_0^2} \quad (78)$$

This is a resistive relationship. Thus, for low-frequency waves, the series impedance, z , in Equation 22 will be resistive rather than inertial. For high-frequency waves it will be inertial, with inertance per unit length given by Equation 41 (see also Eq. 34).

$$\frac{dF}{dx} = -z_I J$$

$$z_I = i2\pi f \frac{\rho_0}{A_0} \quad (79)$$

where f is the frequency of the sine wave. For fixed total volume flow, as the drag effects spread toward the center of the tube, the volume velocity there increases (making up for the reduction in volume velocity near the walls). This increases the effective inertance of the fluid slightly. For frequencies in the neighborhood of f_c (see Eq. 80) acousticians use $\hat{l} = \frac{4\rho}{3A_0}$.

8.2.1 A Cutoff Frequency for Acoustic Waves in Tubes or Ducts

When

$$f = f_c = \frac{4\eta}{A_0\rho} \quad (80)$$

the magnitudes of the inertial and resistive components of impedance per unit length will be approximately equal. For long tubes, therefore, the assumption that viscous effects can be ignored will be applicable if $f \gg f_c$. If f is comparable to or less than f_c , the effects of viscosity will be severe, and the energy of sound waves in the tube will be dissipated (converted to heat) very quickly. For pure water at 20°C (Table A.1),

TABLE A.1. Properties of pure water.

Temperature (°C)	Density (kg/m ³)	Viscosity (kg/m s)
3.98	1000.0	0.1567
10	999.7	0.1307
15	999.1	0.1139
20	998.2	0.1002
25	997.0	0.0890
30	995.7	0.0798
35	994.1	0.0719

$$f_c \approx \frac{0.0004}{A_0} \quad (81)$$

where the cross-sectional area (A_0) of the tube is given in square meters, and f_c is given in Hz. For a periotic duct whose cross-sectional area is n square millimeters, for example, the cutoff frequency would be approximately $400/n$ Hz. With Equation 81 in hand, it is interesting to contemplate the cross-sectional areas of various ducts and canals in both the auditory and the vestibular sides of the vertebrate inner ear, as well as the canals associated with lateral-line structures and those in the bases of the antennules of crustaceans (e.g., see Lewis 1984; Lewis and Narins 1998).

9. Resonance

9.1 Helmholtz Resonators

When a lumped inertance is connected in series with a lumped capacitance, the resulting impedance is the sum of the impedances of the two elements:

$$Z_s(\omega) = i\omega l + \frac{1}{i\omega C} = \frac{1 - \omega^2 lC}{i\omega C} \quad (82)$$

Similarly, when a lumped inertance is connected in parallel with a lumped capacitance, the resulting admittance is the sum of the admittances of the two elements:

$$Y_p(\omega) = i\omega C + \frac{1}{i\omega l} = \frac{1 - \omega^2 lC}{i\omega l} \quad (83)$$

Z_s and Y_p approach 0 as ω approaches $1/\sqrt{lC}$. This means that their reciprocals, Y_s and Z_p , approach infinity under the same circumstances. This critical value of ω is the resonant frequency, ω_r ,

$$\omega_r = \frac{1}{\sqrt{lC}} \quad (84)$$

A commonly cited acoustic example is a volume of air enclosed in a rigid-walled container, connected to the rest of the acoustic circuit through a short, small-diameter, rigid-walled air-filled tube. This forms a series lC circuit, with the values of l and C given by Equations 9 and 12, respectively:

$$\omega_r = \sqrt{\frac{V_{01}\rho_0}{A_{01}^2 c_0^2} \frac{c_0 n_{02}}{\gamma P_0}} = \sqrt{\frac{L_{01} V_{02}}{A_{01}} \frac{\rho_0}{\gamma P_0}} \quad (85)$$

where the subscript 01 refers to the tube, and the subscript 02 refers to the enclosed volume of air. For short tubes, acousticians sometimes apply a correction to account for the momentum carried beyond the end of the tube, re-

placing the tube length, L_{01} , with $L_{01} + 1.7R_{01}$, where R_{01} is the tube radius. This configuration, known as a *Helmholtz resonance*, is used in sound production by animals (Ewing 1989) and some musical instruments (whistle, ocarina, body of violin; see Olson 1967).

Imagine that, beginning at some initial time ($t = 0$) a constant-amplitude, sinusoidal sound pressure waveform at frequency ω , were applied (from a source of some sort) at the mouth of the tube (the end of the tube opposite the enclosed volume). The corresponding volume flow through the tube, into the enclosed volume, would approximate a ramp-modulated sine wave (e.g., $t\sin\omega t$). Cycle by cycle, the amplitude of the sinusoidal volume flow of air through the tube would grow. Correspondingly, the amplitude of the sinusoidal pressure variations in the enclosed volume would grow. Those pressure variations would be precisely 180 degrees out of phase with the pressure variations at the source (at the other end of the tube). Thus the two pressure variations would reinforce one another in the creation of a net pressure difference from one end of the tube to the other. It is that net pressure difference that drives the flow through the tube. The growing flow through the tube and pressure in the enclosure represent a growing amount of acoustic energy stored in the resonant system. In the idealized example presented here, with no mechanism for dissipation of acoustic energy, it would continue to accumulate forever; sinusoidal steady state would never occur. In a real situation, at least two things will happen to the acoustic energy: (1) some fraction of it will be converted to heat by viscous resistance in the system, and (2) some fraction will be radiated from the mouth of the tube and the walls of the enclosure. When the total amount of acoustic energy gained from the source during each cycle is balanced precisely by the acoustic energy lost during each cycle, then sinusoidal steady state will have been achieved.

9.2 Standing-Wave Resonators

In the Helmholtz resonator, one presumes that the wavelength of sound at the resonant frequency is very long in comparison to the dimensions of the enclosed volume and the tube. When that is not so, another form of resonance—a *standing-wave resonance*—can occur. Consider, for example, a rigid-walled tube filled with air. The acoustical driving-point impedance of the air in the tube, viewed from one end of the tube, is given by Equation 68. If the other end of the tube were sealed, making Z_L exceedingly large, then the driving-point impedance would be approximated very well by the following expression:

$$Z_{dp} = Z_0 \frac{\cos\left(2\pi\frac{L}{\lambda}\right)}{i \sin\left(2\pi\frac{L}{\lambda}\right)} \quad (86)$$

which is zero for all values of wavelength (λ for which the tube length (L) divided by wavelength is an odd multiple of one quarter (e.g., $L/\lambda = 1/4, 3/4, 5/4 \dots$). The inertial effects on the flow of air in the tube extend slightly beyond the open end of the tube. Acousticians correct for this by adding (for each open end) $0.82R$ to L in these expressions, where R is the tube radius (Rayleigh 1926; see Olson 1957). At each one of the corresponding frequencies (see Eq. 69), the acoustic path through the air in the pipe exhibits a resonance. As in the Helmholtz resonator, imagine that, beginning at some initial time ($t = 0$) a constant-amplitude, sinusoidal sound pressure waveform at one of the resonant frequencies were applied at the mouth of the tube (the open end of the tube). The time course of the amplitude of the corresponding sinusoidal flow would be essentially the same as that of the Helmholtz resonator, growing linearly with time in the idealized tube. Within the tube would be a standing wave whose amplitude also increases linearly with time. Sinusoidal steady-state conditions would never be achieved. In a real tube, the energy of the standing wave would continue to grow until the energy lost to viscous resistance and acoustic radiation during each cycle of the sound waveform was precisely equal to the energy provided during each cycle by the sound pressure source driving the tube. Sinusoidal steady state would be achieved when that energy balance was achieved.

Standing-wave resonances occur in a wide range of mechanical systems, including flexible strings under tension, rigid bars, stretched membranes, and rigid circular plates (e.g., see Olson 1957). The resonances in strings and bars are easily identified by appropriate application of the transmission-line model.

10. Active Transducers and Amplifiers

A circuit model can be used to represent the flow of free energy from physical realm to physical realm and from place to place within any given physical realm. For a model of the auditory periphery, one form of free energy that would be of interest would be that which begins as the free energy being carried by an acoustical input signal (e.g., a tone), representing a stimulus applied to the external ear or its piscine equivalent. Any element of the model that merely stores that energy, dissipates that energy, or transfers that energy from one place to another or from one physical realm to another is, by definition, *passive*. Circuit models also may include elements in which free energy that originated in the input signal is used to gate the flow of free energy into the model from another source (e.g., a battery of some kind). These often are represented as two-port elements. The free energy that originated in the input signal enters port 1. The gated energy from the battery flows out of port 2. If ports 1 and 2 are represented in the model as being in the same physical realm (e.g., both in the pneumatic realm), then the element is an (*active*) *amplifier*. If the two ports are depicted as being in different physical realms, then the element is an *active transducer*.

10.1 Passivity Implies Bidirectionality

The active nature of the two-port element is reflected in its parameters. Among other things, a passive, linear process must be reciprocal or antireciprocal. Lack of reciprocity or antireciprocity in a linear two-port element implies that the element is active. As a corollary of the Onsager reciprocity theorem, for example, one can state the following: if a transduction process (1) is not based on spinning elements (e.g., a spinning top or rotating charge), (2) operates linearly (e.g., over a small range about the origin), and (3) is not bidirectional, then it *must be active*. Linear, passive transduction processes based on spinning elements are antireciprocal ($h_{12} = h_{21}$, $z_{12} = -z_{21}$) (see Lewis 1996). To be passive, not only must a linear transduction process be bidirectional, it must be either reciprocal or antireciprocal (equally effective in both directions).

10.2 Gated Channels Are Active

On the other hand, gated channels are abundant in nervous systems. In circuit models, the elements representing gated channels would be active. The mechanical input to a stereovillar bundle in an inner-ear auditory sensor carries free energy that originated with the acoustical input to the external ear. This input modulates the opening and closing of the strain-gated channels, through which flows electric current driven by the endolymphatic voltage. The source of the free energy carried by the resulting electrical signal (hair cell receptor potential) is the endolymph battery, not the acoustical input to the ear. Acoustically derived energy modulates the flow of electrical energy from a battery—a quintessential active transduction process. In some instances the hair cell receptor potential, in turn, is applied to a transducer that produces force and motion in the stereovillar bundle (see Manley and Clack, Chapter 1, section 5.1). It is not clear, yet, whether this process is passive, involving merely the transfer of free energy from the receptor potential to the mechanical motion of the bundle, or active, involving the gating of energy from some other source, such as an adenosine triphosphate (ATP) battery, to the motion of the bundle.

10.3 Positive Feedback Can Undamp a Resonance for Sharper Tuning

Whether or not the reverse transduction process (from receptor potential to stereovillar motion) is active, the forward transduction process (from stereovillar motion to receptor potential) definitely is. Thus there is a loop in which there is the possibility of power gain. It has been proposed that such gain could compensate, in part, for the power loss that must be imposed by the viscosity of the water surrounding the stereovillar bundle (see Manley and Clack, Chapter 1, section 5). Assuming that tuning of the hair cell is accomplished by mechanical resonance associated with the stereovillar bundle, for example, one can depict the essential ingredients of this proposition with the circuit model of

Figure A.4. Here the mechanical side is depicted on the left, the electrical side on the right. F_{in} is the force applied to the apex of the bundle by the acoustic input signal; Z_b is the mechanical impedance of the bundle. The upper two-port element represents an idealized active forward transducer [zero driving-point impedance into port 1 (left), zero driving-point admittance into port 2 (right), zero coupling from port 2 to port 1]. The lower two-port element represents an idealized active reverse transducer [zero driving-point admittance into port 2 (right), zero driving-point impedance into port 1 (left), zero coupling from port 1 to port 2]. The hair cell electrical properties are represented by an admittance, Y_m , which relates the receptor potential, F_m , to the receptor current, J_m . The translational motion (velocity J_b) of the apex of the stereovillar bundle is applied through the mechanical sides of both the forward and reverse transducers. For that reason, the corresponding ports are represented as being in series. This in turn means that the force (F_{fb}) developed by the reverse transducer is depicted as being subtracted from the input force:

$$\begin{aligned}
 J_b &= \frac{F_{in} - F_{fb}}{Z_b} \\
 J_m &= -k_b(i\omega)J_b \\
 F_m &= \frac{J_m}{Y_m} = \frac{-k_b(i\omega)J_b}{Y_m} \\
 F_{fb} &= k_m(i\omega)F_m = \frac{-k_m(i\omega)k_b(i\omega)J_b}{Y_m} \\
 F_{in} &= Z_b J_b + F_{fb} = \left(Z_b - \frac{k_m(i\omega)k_b(i\omega)}{Y_m} \right) J_b
 \end{aligned} \tag{87}$$

If it were a simple resonance, the impedance of the bundle (to very small translational motion of its apex) could be written as follows:

$$Z_b = i\omega I_b + \frac{1}{i\omega C_b} + R_b \tag{88}$$

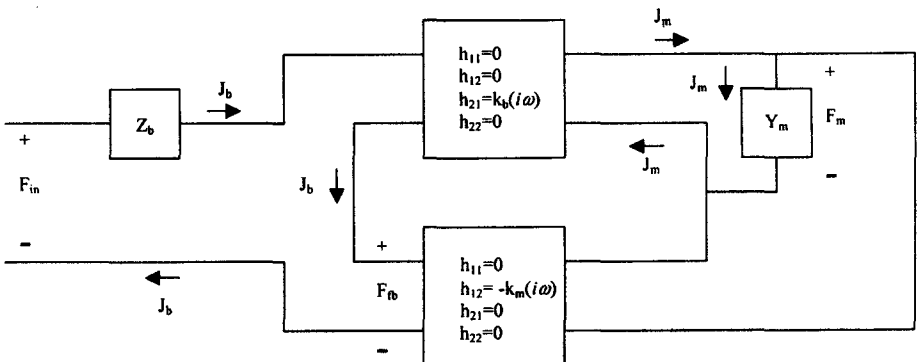


FIGURE A.4. Circuit model depicting forward and reverse transduction in a hair cell.

where I_b is the inertance of the bundle, C_b is its elastic compliance, and R_b is the resistance of the bundle (owing at least in part to the viscosity of water). With the transducers in place, the relationship between bundle motion (velocity) and input force becomes

$$J_b = \frac{F_b}{Z_b - \frac{k_m(i\omega)k_b(i\omega)}{Y_m}} = \frac{F_b}{i\omega I_b + \frac{1}{i\omega C_b} + R_b - \frac{k_m(i\omega)k_b(i\omega)}{Y_m}} \quad (89)$$

The net resistance, which damps the resonance, is equal to

$$R_b - \operatorname{Re} \left\{ \frac{k_m k_b}{Y_m} \right\} \quad (90)$$

To be effective for undamping the resonance, the quantity $\operatorname{Re} \left\{ \frac{k_m k_b}{Y_m} \right\}$ should be greater than zero and less than R_b . If it is greater than R_b , the resonance will oscillate spontaneously. If it remains in the proper range, the pair of active transducers will operate as a stable amplifier (Manley and Clack, Chapter 1, section 5). The effect of this on tuning is depicted in Figure A.5, which shows the phase and amplitude tuning curves of a damped resonance. The bottom panel in Figure A.5 shows the temporal response (e.g., velocity) of the resonance as its residual energy decays after the input has ceased. Notice that as the resonance becomes less damped, its tuning peak becomes sharper but it requires more time to rid itself of residual energy. The latter reflects both the advantage and the disadvantage of resonance tuning: It allows the response to an ongoing input signal to accumulate over time. In a quiet world, where the threshold of hearing is limited by the internal noise of the ear, this should enhance the ability of the hearer to detect the presence of the signal. At the same time, it would reduce the temporal resolution with which the hearer could follow amplitude changes in the signal, once the signal has been detected.

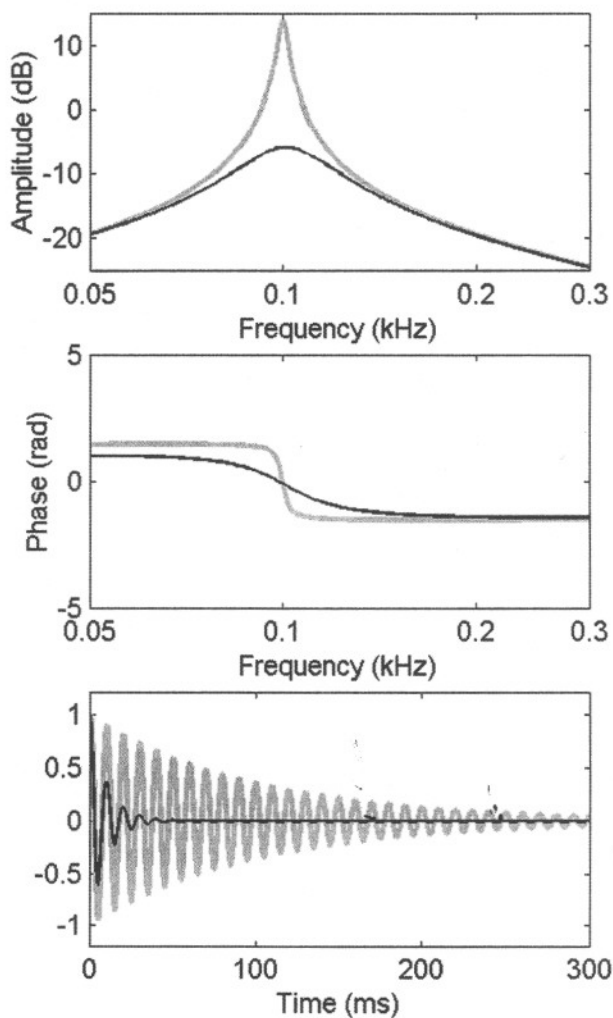


FIGURE A.5. Tuning of a 100-Hz resonance with a high degree of damping (solid black line) and a reduced degree of damping (thick gray line). The bottom panel shows the time course of decay for residual excitation in the resonance.

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