Efficient estimation of physical systems with applications to urban transportation

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December 2, 2014
Outline

Introduction and motivation

Forward problem: traffic assignment

Inverse problem with full data

Inverse problem with missing data

Theoretical results and implementation

Ongoing and future works
Negative impacts of traffic (Texas A&M)

- 1 hour of extra delay for a commute of 20 min without traffic.
- 56 G pounds of CO2 emissions and wasted 2.9 G gallons of fuel.
- Nation’s financial cost: $121 billion.
Limited sensing infrastructure
Limited sensing infrastructure

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Expected life</th>
<th>Cost/lane/year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductive loop detector</td>
<td>10 years</td>
<td>$746</td>
</tr>
<tr>
<td>Video image processor</td>
<td>10 years</td>
<td>$580</td>
</tr>
</tbody>
</table>

Middleton and Parker. *Initial Evaluation of Selected Detectors to Replace Inductive Loops on Freeways*, FHWA/TX-00/1439-7. Texas Transportation Institute, College Station, TX. April 2000.
Sparsity of the data

20476 links (OSM)
1033 observed (PeMS)
~ 95% missing data
Efficient estimation in urban transportation: previous works

Highway Traffic estimation

\[ x_{i-1} \quad x_i \quad x_{i+1} \]

Problem

Cell Transmission model

Technique

Kalman filter

Hybrid estimation
Efficient estimation in urban transportation: previous works

Highway Traffic estimation

Fusion of loop & cellular data for estimation

Problem

- Cell Transmission model
- L1 constraints

Technique

- Hybrid estimation
- Constraint elimination
- Isotonic regression

Problem

- Hybrid estimation
- Constraint elimination
- Isotonic regression

Introduction and motivation
Efficient estimation in urban transportation: previous works

Highway Traffic estimation

Fusion of loop & cellular data for estimation

Estimation & control of the traffic network in equilibrium

Introduction and motivation
Quasi-static traffic assignment problem
Quasi-static traffic assignment problem

Introduction and motivation
Quasi-static traffic assignment problem

Specify delay function on each arc:

Wardrop equilibrium
Common solution concept for traffic models: each agent has access to the delay function on each arc and chooses the shortest path from origin to destination.
We pose the inverse traffic assignment problem with missing data
Problem statement

We pose the inverse traffic assignment problem with missing data

- Traffic volumes resulting from rational behavior of agents on the road network are easily but sparsely observable.
We pose the inverse traffic assignment problem with missing data

- Traffic volumes resulting from rational behavior of agents on the road network are easily but **sparsely observable**.

- Delay functions are not directly observable.
We pose the inverse traffic assignment problem with missing data

- Traffic volumes resulting from rational behavior of agents on the road network are easily but *sparsely observable*.

- Delay functions are not directly observable.

- How can we impute the delay functions from *partial observations* of equilibria?
Previous works assume full observations

Inverse convex optimization

Inverse variational inequality
Previous works assume full observations

Inverse convex optimization

Inverse variational inequality

Previous works:

Our work:

- Inverse problem with full data → Convex optimization problem
- Inverse problem with missing data → Bilevel program
Based on partial observations, we want to impute...


Traffic assignment and potential games

Our work combines ideas from and contribute to...

**Bilevel programming and Mathematical Programs with Equilibrium Constraints**


**Computational mathematics**


**Pareto optimization**

Outline and contributions

Forward problem: Traffic assignment problem
Outline and contributions

Forward problem: Traffic assignment problem \rightarrow \text{Nash} \rightarrow \text{Variational Inequality (VI)}
Outline and contributions

Forward problem: Traffic assignment problem

Nash

Variational Inequality (VI)

Beckmann

Convex Optimization (CO)
Outline and contributions

Forward problem:
- Traffic assignment problem
  - Nash
  - Variational Inequality (VI)
  - Convex Optimization (CO)

Reverse problem:
- Inverse VI/CO with full data

Introduction and motivation
Outline and contributions

Forward problem:
- Traffic assignment problem
  - Nash
  - Variational Inequality (VI)
  - Convex Optimization (CO)

Reverse problem:
- Inverse VI/CO with full data
  - Pareto optimization
  - Inverse VI/CO with missing data
Outline and contributions

Forward problem:
- Traffic assignment problem
  - Nash
  - Variational Inequality (VI)
  - Beckmann
  - Convex Optimization (CO)

Reverse problem:
- Inverse VI/CO with full data
  - Pareto optimization
- Inverse VI/CO with missing data
  - Approx. bilevel program
- Pareto optimization

Introduction and motivation
Outline and contributions

Forward problem:
- Traffic assignment problem
- Nash
- Variational Inequality (VI)
- Convex Optimization (CO)

Reverse problem:
- Inverse VI/CO with full data
  - Pareto optimization
  - Computational mathematics
  - Theoretical bounds on residuals
- Inverse VI/CO with missing data
- Approximate bilevel program
Outline and contributions

Forward problem:

- Traffic assignment problem
- Nash
- Variational Inequality (VI)
- Convex Optimization (CO)

Reverse problem:

- Inverse VI/CO with full data
- Pareto optimization
- Theoretical bounds on residuals
- Solution algorithm
- Inverse VI/CO with missing data
- Approx. bilevel program
- Pareto optimization

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Inverse problem with missing data

Theoretical results and implementation

Future works
Outline

Introduction and motivation

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Theoretical results and implementation

Ongoing and future works
Morning commute example for the traffic assignment problem

- $\mathcal{A} = \text{arc set} = \{a, b, c, d, e\}$
- $\mathcal{N} = \text{node set} = \{1, 2, 3, 4\}$
- Commodity 1: $c_1 = (1 \rightarrow 4, 1000)$ "routing 1000 veh/h from 1 to 4"
- Commodity 2: $c_2 = (2 \rightarrow 4, 2000)$ "routing 2000 veh/h from 2 to 4"
- $\mathcal{C} = \text{commodity set} = \{c_1, c_2\}$
Morning commute example for the traffic assignment problem

\( A = \{a, b, c, d, e\}, \ |A| = 5, \ C = \{1 \rightarrow 4, 2 \rightarrow 4\}, \ |C| = 2 \)

- commodity flow vectors: \( x_1 \in \mathbb{R}^{|A|}, \ x_2 \in \mathbb{R}^{|A|} \)
- overall flow vector: \( x = (x_1, x_2) \in \mathbb{R}^{|A| \times |C|} \)
- aggregate flow vector: \( v = x_1 + x_2 \in \mathbb{R}^{|A|} \)

Flow conservation for commodity 1
- \( x_a - x_b = 1000 \)
- \( x_a - x_c - x_e = 0 \)
- \( x_b + x_e - x_d = 0 \)
- \( x_c + x_d = 1000 \)
nonnegative flows

\( \Rightarrow N \ x_1 = b_1 \)
\( x_1 \geq 0 \)

Flow conservation for commodity 2
- \( x_a - x_b = 0 \)
- \( x_a - x_c - x_e = -2000 \)
- \( x_b + x_e - x_d = 0 \)
- \( x_c + x_d = 2000 \)
nonnegative flows

\( \Rightarrow N \ x_2 = b_2 \)
\( x_2 \geq 0 \)
Morning commute example for the traffic assignment problem

Feasible set:

$$\mathcal{K} = \{ \mathbf{x} = (x_1, x_2) \mid Nx_1 = b_1, x_1 \geq 0, Nx_2 = b_2, x_2 \geq 0 \}$$

Delay map $S : \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|}$ w.r.t. aggregate flow $\mathbf{v} = x_1 + x_2$

$$S(\mathbf{v}) = (s_a(v_a), s_b(v_b), s_c(v_c), s_d(v_d), s_e(v_e)) \in \mathbb{R}^{|A|}$$

$x^* \in \mathcal{K}$ is a Nash eq. if $\forall x \in \mathcal{K}$, the associated aggregate flows $\mathbf{v}^*, \mathbf{v}$ are such that

$$v_a^* s_a(v_a^*) + \cdots + v_e^* s_e(v_e^*) \leq v_a s_a(v_a^*) + \cdots + v_e s_e(v_e^*)$$

Forward problem: traffic assignment
General traffic assignment problem

- Commodity flow vectors: $x_k \in \mathbb{R}^{|A|}$ for all $k \in C$

Forward problem: traffic assignment
General traffic assignment problem

- commodity flow vectors: \( x_k \in \mathbb{R}^{|A|} \) for all \( k \in C \)
  - overall flow vector: \( \mathbf{x} = (x_k)_{k \in C} \in \mathbb{R}^{|A| \times |C|} \)

\( S: \mathbb{R}^{|A|} \to \mathbb{R}^{|A|} \) such that \( S(v) = (s_a(v_a))_{a \in A} \)

Nash equilibrium: \( \mathbf{x}^* \in K \) such that for all \( \mathbf{x} \in K \)
\[ \sum_{a \in A} v^*_{a} S_{a}(v^*) \leq \sum_{a \in A} v_{a} S_{a}(v) \iff S(v^*)^T v^* \leq S(v)^T v \iff F(x^*)^T x^* \leq F(x)^T x \]
General traffic assignment problem

- commodity flow vectors: \( \mathbf{x}_k \in \mathbb{R}^{|A|} \) for all \( k \in C \)
- overall flow vector: \( \mathbf{x} = (\mathbf{x}_k)_{k \in C} \in \mathbb{R}^{|A| \times |C|} \)
- aggregate flow vector: \( \mathbf{v} = \sum_{k \in C} \mathbf{x}_k = \mathbf{Z} \mathbf{x} \in \mathbb{R}^{|A|} \)
General traffic assignment problem

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- overall flow vector: \( x = (x_k)_{k \in C} \in \mathbb{R}^{|A| \times |C|} \)
- aggregate flow vector: \( v = \sum_{k \in C} x_k = Zx \in \mathbb{R}^{|A|} \)
- feasible set: \( K = \{ x = (x_k)_{k \in C} \mid N x_k = b_k, x_k \geq 0, \forall k \in C \} \)
  \[ = \{ x \in \mathbb{R}^{|A| \times |C|} \mid A x = b, x \succeq 0 \} \]
General traffic assignment problem

- commodity flow vectors: \( x_k \in \mathbb{R}^{|A|} \) for all \( k \in \mathcal{C} \)
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- aggregate flow vector: \( \mathbf{v} = \sum_{k \in \mathcal{C}} x_k = \mathbf{Zx} \in \mathbb{R}^{|A|} \)
- feasible set: \( \mathcal{K} = \{ \mathbf{x} = (x_k)_{k \in \mathcal{C}} | \mathbf{Nx}_k = \mathbf{b}_k, x_k \geq 0, \forall k \in \mathcal{C} \} \)
  \( = \{ \mathbf{x} \in \mathbb{R}^{|A| \times |\mathcal{C}|} | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0 \} \)
- Delay map \( S : \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|} \) such that \( S(\mathbf{v}) = (s_a(v_a))_{a \in \mathcal{A}} \)
General traffic assignment problem

- commodity flow vectors: $x_k \in \mathbb{R}^{|A|}$ for all $k \in C$
- overall flow vector: $\mathbf{x} = (x_k)_{k \in C} \in \mathbb{R}^{|A| \times |C|}$
- aggregate flow vector: $\mathbf{v} = \sum_{k \in C} x_k = Z\mathbf{x} \in \mathbb{R}^{|A|}$
- feasible set: $\mathcal{K} = \{ \mathbf{x} = (x_k)_{k \in C} \mid N\mathbf{x}_k = b_k, x_k \succeq 0, \forall k \in C \}$
  $$= \{ \mathbf{x} \in \mathbb{R}^{|A| \times |C|} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq 0 \}$$

- Delay map $S : \mathbb{R}^{|A|} \to \mathbb{R}^{|A|}$ such that $S(\mathbf{v}) = (s_a(\mathbf{v}_a))_{a \in A}$
- Nash equilibrium: $\mathbf{x}^* \in \mathcal{K}$ such that for all $\mathbf{x} \in \mathcal{K}$
  $$\sum_{a \in A} v_a^* s_a(\mathbf{v}_a^*) \leq \sum_{a \in A} v_a s_a(\mathbf{v}_a^*) \iff S(\mathbf{v}^*)^T \mathbf{v}^* \leq S(\mathbf{v}^*)^T \mathbf{v}$$
  $$\iff S(Z\mathbf{x}^*)^T Z\mathbf{x}^* \leq S(Z\mathbf{x}^*)^T Z\mathbf{x}$$
  $$\iff F(\mathbf{x}^*)^T \mathbf{x}^* \leq F(\mathbf{x}^*)^T \mathbf{x}$$

$\implies$ Nash eq. = solution to a VI with $F(\mathbf{x}) = Z^T S(Z\mathbf{x})$
General traffic assignment problem

- commodity flow vectors: \( \mathbf{x}_k \in \mathbb{R}^{|A|} \) for all \( k \in C \)
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- Delay map \( S : \mathbb{R}^{|A|} \to \mathbb{R}^{|A|} \) such that \( S(\mathbf{v}) = (s_a(\mathbf{v}_a))_{a \in A} \)
- Nash equilibrium: \( \mathbf{x}^* \in \mathcal{K} \) such that for all \( \mathbf{x} \in \mathcal{K} \)
  \[\sum_{a \in A} \mathbf{v}_a^* \cdot s_a(\mathbf{v}_a^*) \leq \sum_{a \in A} \mathbf{v}_a \cdot s_a(\mathbf{v}_a^*) \iff S(\mathbf{v}^*)^T \mathbf{v}^* \leq S(\mathbf{v}^*)^T \mathbf{v} \]
  \[\iff S(\mathbf{Zx}^*)^T \mathbf{Zx}^* \leq S(\mathbf{Zx}^*)^T \mathbf{Zx} \]
  \[\iff F(\mathbf{x}^*)^T \mathbf{x}^* \leq F(\mathbf{x}^*)^T \mathbf{x} \]
  \[\implies \text{Nash eq. = solution to a VI with } F(\mathbf{x}) = \mathbf{Z}^T S(\mathbf{Zx}) \]

Definition: variational inequality (VI)

\( \text{VI(} \mathcal{K}, F) \): find \( \mathbf{x}^* \in \mathcal{K} \) such that \( F(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in \mathcal{K} \).
Optimization process and Variational inequality

Theorem 1 (Beckmann et al. 1956)

Suppose the arc delay functions are nonnegative, continuous, monotone, separable. Then the Nash equilibrium is solution of a convex optimization program, denoted $\text{OP}(\mathcal{K}, f)$

$$\min f(x) \quad \text{s.t.} \quad Ax = b, \ x \succeq 0$$

Remarks

- The potential $f$ encodes the interaction between players.
- $\mathcal{K} := \{x | Ax = b, \ x \succeq 0\}$ encodes the flow conservation.
Theorem 1 (Beckmann et al. 1956)

Suppose the arc delay functions are nonnegative, continuous, monotone, separable. Then the Nash equilibrium is solution of a convex optimization program, denoted $\text{OP}(\mathcal{K}, f)$

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$$

Remarks

- The potential $f$ encodes the interaction between players.
- $\mathcal{K} := \{x \mid Ax = b, \ x \succeq 0\}$ encodes the flow conservation.

Theorem 2

With $f \in C^1$, $x^* \in \mathcal{K}$ is solution iff $\nabla f(x^*)^T(u - x^*) \geq 0, \ \forall u \in \mathcal{K}$.

Result from Beckmann: for the map $F(x) = Z^T S(Zx)$, $\exists f$ convex such that $F = \nabla f$
Outline

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Theoretical results and implementation

Ongoing and future works
Review of Inverse problem

Equilibrium model

$$(K, F(\cdot))$$

Strategy set $K$

$K \subset \mathbb{R}^n$ closed convex

Payoffs function $F$

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

Set of equilibria $S \subset \mathbb{R}^n$

strategy vector $x \in S$

equilibrium model observations with missing data

parametric model

guess on the inputs of the models
Review of Inverse problem

\( (K, F(\cdot)) \)

Strategy set \( K \subset \mathbb{R}^n \) closed

Payoffs function \( F : \mathbb{R}^n \to \mathbb{R}^n \)

Set of equilibria \( S \subset \mathbb{R}^n \)

strategy vector \( x \in S \)

\[ S \text{ is the set solution of } VI(K, F) \]

Variational inequality

\[ VI(K, F) : \text{ Find } x \in K \text{ such that } F(x)^T(u - x) \geq 0 \quad \forall u \in K \]

Feasible Set \( K \)

\[ F(x^*) \]

\[ u - x^* \]

Figure adapted from [Scutari2010]

equilibrium model \hspace{2cm} \text{observations with missing data} \hspace{2cm} \text{parametric model} \hspace{2cm} \text{guess on the inputs of the models}

Inverse problem with full data
Review of Inverse problem

**Equilibrium model**

\((K, F(\cdot))\)

- Strategy set \(K \subset \mathbb{R}^n\) closed convex
- Payoffs function \(F: \mathbb{R}^n \rightarrow \mathbb{R}^n\)
- Set of equilibria \(S \subset \mathbb{R}^n\)
- Strategy vector \(x \in S\)

**Parametric model**

Assumes a parametric model \((K, F(\cdot|\theta)), \theta \in \Theta\)

- Structural parameters \(\theta \in \mathbb{R}^d\)
- \(\Theta\) contains enough prior information about the model

observes \(z = Hx \in \mathbb{R}^p\)

with \(x \in S\) and \(H\) the observer

guess on the inputs of the models
Review of Inverse problem

**Equilibrium model**

(K, F(·))

Strategy set $K \subseteq \mathbb{R}^n$ closed convex

Payoffs function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

Set of equilibria $S \subseteq \mathbb{R}^n$

strategy vector $x \in S$

**Parametric model**

Assumes a parametric model

$(K, F(·|\theta))$, $\theta \in \Theta$

Structural parameters $\theta \in \mathbb{R}^d$

$\Theta$ contains *enough* prior information about the model

observes $z = Hx \in \mathbb{R}^p$

with $x \in S$ and $H$ the observer

---

**Mathematical program**

$$\min_{x, \theta} \quad ||Hx - z||$$

s.t. $x \in S(\theta)$

$\theta \in \Theta$

Find the best $\theta \in \Theta$ to

minimize the measurement residual $||Hx - z||$

such that $x$ is an equilibrium of $F(K, F(·|\theta))$,
Primal-dual system and KKT conditions

Assumption: \( \mathcal{K} = \{ \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq 0 \} \)

Theorem 3: primal-dual system (Facchinei 2003 Aghassi 2005)

\( \mathbf{x} \) is solution to \( \text{VI}(\mathcal{K}, \mathbf{F}) \) if and only if there exists \( \mathbf{y} \) such that

\[
\mathbf{F}(\mathbf{x})^T \mathbf{x} = \mathbf{b}^T \mathbf{y} \\
\mathbf{A}\mathbf{x} = \mathbf{b} \\
\mathbf{x} \succeq 0 \text{ primal feasibility} \\
\mathbf{A}^T \mathbf{y} \succeq \mathbf{F}(\mathbf{x}) \text{ dual feasibility}
\]

Theorem 4: KKT conditions (Harker 1989)

\( \mathbf{x} \) is solution to \( \text{VI}(\mathcal{K}, \mathbf{F}) \) if and only if there exists \( (\mathbf{y}, \pi) \) such that

\[
\mathbf{F}(\mathbf{x}) = \mathbf{A}^T \mathbf{y} + \pi \mathbf{A}\mathbf{x} = \mathbf{b} \\
\mathbf{x} \succeq 0 \text{ primal feasibility} \\
\pi \succeq 0, \mathbf{x}^T \pi = 0 \text{ dual feasibility}
\]

Note: If \( \mathbf{F} = \nabla f \), we can substitute \( \text{VI}(\mathcal{K}, \mathbf{F}) \) with \( \text{OP}(\mathcal{K}, f) \).
Primal-dual system and KKT conditions

Assumption: \( \mathcal{K} = \{ x \mid A x = b, \ x \succeq 0 \} \)

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\[
\begin{align*}
F(x)^T x &= b^T y \\
A x &= b, \ x \succeq 0 \\
A^T y &\preceq F(x)
\end{align*}
\]

**Theorem 4:** KKT conditions (Harker 1989)

\( x \) is solution to \( \text{VI}(\mathcal{K}, F) \) if and only if there exists \((y, \pi)\) such that

\[
\begin{align*}
F(x) &= A^T y + \pi A x = b \\
x \succeq 0 \text{ primal feasibility} \\
\pi \succeq 0, \ x^T \pi = 0 &\text{ dual feasibility}
\end{align*}
\]

Note: If \( F = \nabla f \), we can substitute \( \text{VI}(\mathcal{K}, F) \) with \( \text{OP}(\mathcal{K}, f) \).
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\end{align*}
\]

Theorem 4: KKT conditions (Harker 1989)

\( x \) is solution to \( \text{VI}(\mathcal{K}, F) \) if and only if there exists \((y, \pi)\) such that

\[
\begin{align*}
F(x) &= A^T y + \pi \\
Ax &= b, \ x \succeq 0 \\
\pi \succeq 0, \ x^T \pi &= 0
\end{align*}
\]

Note: If \( F = \nabla f \), we can substitute \( \text{VI}(\mathcal{K}, F) \) with \( \text{OP}(\mathcal{K}, f) \).
Primal-dual system and KKT conditions

Assumption: \( \mathcal{K} = \{ \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq 0 \} \)

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\[ x \text{ is solution to } \text{VI}(\mathcal{K}, F) \text{ if and only if there exists } y \text{ such that} \]

\[
F(x)^T x = b^T y \\
A x = b, \ x \succeq 0 \quad \text{primal feasibility} \\
A^T y \preceq F(x) \quad \text{dual feasibility}
\]

Theorem 4: KKT conditions (Harker 1989)

\[ x \text{ is solution to } \text{VI}(\mathcal{K}, F) \text{ if and only if there exists } (y, \pi) \text{ such that} \]

\[
F(x) = A^T y + \pi \\
A x = b, \ x \succeq 0 \quad \text{primal feasibility} \\
\pi \succeq 0, \ x^T \pi = 0 \quad \text{dual feasibility}
\]

Note: If \( F = \nabla f \), we can substitute \( \text{VI}(\mathcal{K}, F) \) with \( \text{OP}(\mathcal{K}, f) \).
**Primal-dual system and KKT conditions**

**Assumption:** \( \mathcal{K} = \{ x \mid Ax = b, \ x \succeq 0 \} \)

---

### Theorem 3: primal-dual system (Facchinei 2003, Aghassi 2005)

\( x \) is solution to VI(\( \mathcal{K}, F \)) if and only if there exists \( y \) such that

\[
F(x)^T x = b^T y \\
Ax = b, \ x \succeq 0 \quad \text{primal feasibility} \\
A^T y \preceq F(x) \quad \text{dual feasibility}
\]

---

### Theorem 4: KKT conditions (Harker 1989)

\( x \) is solution to VI\( (\mathcal{K}, F) \) if and only if there exists \( (y, \pi) \) such that

\[
F(x) = A^T y + \pi \\
Ax = b, \ x \succeq 0 \quad \text{primal feasibility} \\
\pi \succeq 0, \ x^T \pi = 0 \quad \text{dual feasibility}
\]

---

Note: If \( F = \nabla f \), we can substitute VI\( (\mathcal{K}, F) \) with OP\( (\mathcal{K}, f) \).
Nonnegative functions $r_{PD}$ and $r_{KKT}$ such that

- $r_{PD}(x, y) = 0 \iff (x, y)$ solution to primal-dual system
- $r_{KKT}(x, y, \pi) = 0 \iff (x, y, \pi)$ solution to KKT system
Residual functions

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<td>$r_{PD}(\mathbf{x}, \mathbf{y}) = 0 \iff (\mathbf{x}, \mathbf{y})$ solution to primal-dual system</td>
</tr>
<tr>
<td>$r_{KKT}(\mathbf{x}, \mathbf{y}, \pi) = 0 \iff (\mathbf{x}, \mathbf{y}, \pi)$ solution to KKT system</td>
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</table>

Classic residual associated to the primal-dual system

$$r_{PD}(\mathbf{x}, \mathbf{y}) = F(\mathbf{x})^T \mathbf{x} - \mathbf{b}^T \mathbf{y}$$
Residual functions

**Definition: residual functions**

Nonnegative functions $r_{PD}$ and $r_{KKT}$ such that

\[
r_{PD}(x, y) = 0 \iff (x, y) \text{ solution to primal-dual system}
\]

\[
r_{KKT}(x, y, \pi) = 0 \iff (x, y, \pi) \text{ solution to KKT system}
\]

**Classic residual associated to the primal-dual system**

\[
r_{PD}(x, y) = F(x)^T x - b^T y
\]

**Classic residual associated to the KKT system, for $\alpha > 0$**

\[
r_{KKT}^p(x, y, \pi) = \|\alpha r_{stat} + r_{comp}\|_p
\]

with

\[
r_{stat}(x, y, \pi) = F(x)^T x - A^T y - \pi
\]

\[
r_{comp}(x, \pi) = x \circ \pi = (x_i \pi_i)_{i=1}^n
\]

Inverse problem with full data
Inverse problem with full data

Previous works: estimate the delay functions from full data.

1Bertsimas et al. (2014) and Keshavarz, Wang, and Boyd (2011)
Inverse problem with full data

Previous works: estimate the delay functions from full data.

**Notation**: $\text{MP}(\mathcal{K}, F)$ both refers to $\text{VI}(\mathcal{K}, F)$ and $\text{OP}(\mathcal{K}, f)$

Given $x^{\text{obs}}$ (nearly) optimal for $\text{MP}(\mathcal{K}, F)$, the inverse problem is convex:\(^1\)

$$\min_{y, \theta} \ r(x^{\text{obs}}, y, \theta)$$

s.t. dual feasibility

$$\theta \in \Theta$$

---

\(^1\)Bertsimas et al. (2014) and Keshavarz, Wang, and Boyd (2011)
Inverse problem with full data

Previous works: estimate the delay functions from full data.

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Given $x^{\text{obs}}$ (nearly) optimal for $\text{MP}(\mathcal{K}, F)$, the inverse problem is convex:$^1$

$$\min_{y, \theta} \quad r(x^{\text{obs}}, y, \theta)$$

s.t. dual feasibility

$\theta \in \Theta$

We develop a rigorous mathematical framework for imputing the delay functions from sparse data.

---

$^1$Bertsimas et al. (2014) and Keshavarz, Wang, and Boyd (2011)
Outline

Introduction and motivation

Forward problem: traffic assignment

Inverse problem with full data

Inverse problem with missing data

Theoretical results and implementation

Ongoing and future works
Estimation of the highway network near Los Angeles

Figure: Highway network near Los Angeles

True delay function:
\[ s_{\text{true}}(v_a) = d_a(1 - 3.5 \frac{3}{v_a} + 3.5 \frac{3}{v_a}) \]

Parametric delay:
\[ s_a(v_a | \theta) = d_a(1 + \sum_{i=1}^{6} \theta_i (\frac{v_a}{m_a})^i) \]

where
- \( d_a \) = free flow delay,
- \( m_a \) = number of lanes,
- \( v_a \) = aggregate flow.
Estimation of the highway network near Los Angeles

**True delay function:** \( s_{a}^{\text{true}}(v_a) = d_a \left( 1 - \frac{3.5}{3} + \frac{3.5}{3 - \frac{v_a}{m_a}} \right) \)

**Parametric delay:** \( s_a(v_a|\theta) = d_a \left( 1 + \sum_{i=1}^{6} \theta_i \left( \frac{v_a}{m_a} \right)^i \right) \)

where \( d_a = \) free flow delay, \( m_a = \) number of lanes, \( v_a = \) aggregate flow.

**Figure:** Highway network near Los Angeles

Inverse problem with missing data 28
Estimation of the highway network near Los Angeles

Figure: Delay function imputation.

Figure: Toll pricing.
New formulation of the inverse problem with missing data

**Notation:** $MP(\mathcal{K}, F)$ both refers to $VI(\mathcal{K}, F)$ and $OP(\mathcal{K}, f)$

\[
\min_{y, \theta} \quad r(x^{\text{obs}}, y, \theta) \\
\text{s.t.} \quad \text{dual feasibility} \quad \theta \in \Theta
\]
New formulation of the inverse problem with missing data

**Notation:** $MP(K, F)$ both refers to $VI(K, F)$ and $OP(K, f)$

\[
\begin{align*}
\min_{x, y, \theta} & \quad r(x, y, \theta) \\
\text{s.t.} & \quad \text{dual feasibility} \\
& \quad x = x^{\text{obs}} \\
& \quad \theta \in \Theta
\end{align*}
\]
New formulation of the inverse problem with missing data

**Notation:** $\text{MP}(\mathcal{K}, F)$ both refers to $\text{VI}(\mathcal{K}, F)$ and $\text{OP}(\mathcal{K}, f)$

$$\begin{align*}
\min_{x, y, \theta} & \quad r(x, y, \theta) \\
\text{s.t.} & \quad \text{dual feasibility} \\
& \quad \text{primal feasibility} \\
& \quad Hx = z^{\text{obs}} \\
& \quad \theta \in \Theta
\end{align*}$$
New formulation of the inverse problem with missing data

**Notation:** $\text{MP}(\mathcal{K}, F)$ both refers to $\text{VI}(\mathcal{K}, F)$ and $\text{OP}(\mathcal{K}, f)$

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\min_{x, y, \theta} & \quad r(x, y, \theta) \\
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& \quad \theta \in \Theta
\end{align*}$$

**New formulation in the case of partial observations**

- impose **primal feasibility** on the induced response
- formulation **robust to outliers** in the observations

and $Ax = b$, $Hx = z$ might be infeasible because of noise
New formulation of the inverse problem with missing data

Notation: $\text{MP}(\mathcal{K}, F)$ both refers to $\text{VI}(\mathcal{K}, F)$ and $\text{OP}(\mathcal{K}, f)$

\[
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\min_{x, y, \theta} & \quad r(x, y, \theta) \\
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New formulation in the case of partial observations

- impose **primal feasibility** on the induced response
- formulation **robust to outliers** in the observations
- and $Ax = b$, $Hx = z$ might be infeasible because of noise

\[
\begin{align*}
\min_{x, y, \theta} & \quad [r(x, y, \theta), \phi(Hx - z^{\text{obs}})]^T \\
\text{s.t.} & \quad \text{primal feasibility} \\
\quad & \quad \text{dual feasibility} \\
\quad & \quad \theta \in \Theta
\end{align*}
\]
Novel single-level formulation of bilevel programs

General formulation of the bilevel program

$$\min_{x, \theta} g(x, \theta)$$

s.t. \( x \) is solution to MP\((\mathcal{K}, F(\cdot, \theta))\)

\( \theta \in \Theta \)
Novel single-level formulation of bilevel programs

General formulation of the bilevel program

\[
\begin{align*}
\min_{x, \theta} & \quad g(x, \theta) \\
\text{s.t.} & \quad x \text{ is solution to } \text{MP}(\mathcal{K}, F(\cdot, \theta)) \\
& \quad \theta \in \Theta
\end{align*}
\]

Remarks about bilevel programming:

- Wide class of programs with many applications, Dempe (2002).
- Hard to solve because of the bilevel structure.
- Penalization is common to avoid the complementary conditions

\[x^T \pi, \quad F(x)^T x = b^T y\]
Novel single-level formulation of bilevel programs

General formulation of the bilevel program

\[
\begin{align*}
\min_{x, \theta} & \quad g(x, \theta) \\
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\]

Remarks about bilevel programming:

- Wide class of programs with many applications, Dempe (2002).
- Hard to solve because of the bilevel structure.
- Penalization is common to avoid the complementary conditions

\[
x^T \pi, \quad F(x)^T x = b^T y
\]

With \( \phi \) substituted with general objective \( g \), our formulation is a novel single-level formulation of bilevel programs

\[
\begin{align*}
\min_{x,y,\theta} & \quad [r(x, y, \theta), g(x, \theta)]^T \\
\text{s.t.} & \quad \text{primal feasibility} \\
& \quad \text{dual feasibility} \\
& \quad \theta \in \Theta
\end{align*}
\]
Outline

Introduction and motivation

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Theoretical results and implementation

Ongoing and future works
Bounds on residuals

Theorem 5 (Bertsimas et al. 2014)

Suppose primal feasibility and dual feasibility hold. Then
\[ r_{PD} \leq \epsilon \iff r_{VI} \leq \epsilon \implies r_{OP} \leq \epsilon \]

Theorem 6 (Thai and Bayen 2014)

Suppose primal and dual feasibilities hold. Then \( \forall p \geq 1, \alpha > 0 \)
\[ r_{VI} \leq \epsilon \implies r_{KKT}^p \leq \epsilon \]
Reciprocally,
when \( p > 1 \), \[ r_{KKT}^p \leq \epsilon \implies r_{VI} \leq \epsilon \left( 1 + \left( \|x\|_{\infty}/\alpha \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} n^{1-\frac{1}{p}} \]
when \( p = 1 \), \[ r_{KKT}^p \leq \epsilon \implies r_{VI} \leq \epsilon \left( 1 + \left( \|x\|_{\infty}/\alpha - 1 \right)_+ \right) \]

- Tight bounds.
- Asymptotically, \( r_{KKT}^p \leq \epsilon \implies r_{VI} = O(\epsilon \|x\|_{\infty} n^{1-\frac{1}{p}}) \).
- \( r_{VI} \) and \( r_{KKT} \) define different metrics.
- \textit{i.e.} a model can fit the data under \( r_{KKT} \) but not under \( r_{PD} \).
Bounds on residuals for strongly monotone functions

Definition: strong monotonicity

A map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is strongly monotone if $\exists m > 0$ such that
\[
(F(x) - F(y))^T (x - y) \geq m \|x - y\|^2_2, \quad \forall x, y \in K
\]

- equivalent to strong convexity of $f$ when $\nabla f = F$.
- unique solution $x^*$ to $\text{VI}(K, F)$, resp. $\text{OP}(K, f)$.
Bounds on residuals for strongly monotone functions

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- equivalent to strong convexity of $f$ when $\nabla f = F$.
- unique solution $x^*$ to $\text{VI}(\mathcal{K}, F)$, resp. $\text{OP}(\mathcal{K}, f)$.

Theorem 7 (Pang 1996)

Suppose $F$ strongly monotone, primal and dual feasibilities, then

$$r_{PD} \leq \epsilon \implies \|x - x^*\|_2 \leq \sqrt{\epsilon/m}$$

Theoretical results and implementation
Bounds on residuals for strongly monotone functions

**Definition:** strong monotonicity

A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone if $\exists m > 0$ such that

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- equivalent to strong convexity of $f$ when $\nabla f = F$.
- unique solution $x^*$ to VI($\mathcal{K}, F$), resp. OP($\mathcal{K}, f$).

**Theorem 7 (Pang 1996)**

Suppose $F$ strongly monotone, primal and dual feasibilities, then

$$r_{PD} \leq \epsilon \implies \|x - x^*\|_2 \leq \sqrt{\epsilon / m}$$

**Theorem 8 (Thai and Bayen 2014)**

Suppose $F$ strongly monotone, primal and dual feasibilities, then

$$r_{KKT} \leq \epsilon \implies \|x - x^*\|_2 \leq O\left(\sqrt{\epsilon \|x\|_\infty n^{1-\frac{1}{p}} / m}\right)$$
Finding the optimal Pareto point in one shot

\[
\begin{align*}
\min_{x,y,\theta} & \quad \left[ r(x, y, \theta), \phi(Hx - z^{obs}) \right]^T \\
\text{s.t.} & \quad Ax = b, \ x \succeq 0 \\
& \quad A^T y \preceq F(x|\theta) \\
& \quad \theta \in \Theta
\end{align*}
\]
Finding the optimal Pareto point in one shot

$$\min_{x, y, \theta} w_{mp} r(x, y, \theta) + w_{obs} \phi(Hx - z^{obs})$$

s.t.  
$$Ax = b, \ x \geq 0$$
$$A^T y \preceq F(x|\theta)$$
$$\theta \in \Theta$$
Finding the optimal Pareto point in one shot

$$\begin{align*}
\min_{x,y,\theta} & \quad w_{mp} (F(x|\theta)^T x - b^T y) + w_{obs} \|Hx - z^{obs}\|_2^2 \\
\text{s.t.} & \quad Ax = b, \ x \succeq 0 \\
& \quad A^T y \preceq F(x|\theta) \\
& \quad \theta \in \Theta
\end{align*}$$

Theoretical results and implementation
Finding the optimal Pareto point in one shot

\[
\min_{x, y, \theta} \quad w_{\text{mp}} \ r(x, y, \theta) + w_{\text{obs}} \ \phi(Hx - z^{\text{obs}})
\]
\[
\text{s.t.} \quad Ax = b, \ x \geq 0
\]
\[
A^T y \preceq F(x|\theta)
\]
\[
\theta \in \Theta
\]

Classic methodology to explore the Pareto curve

1. Normalize: \( \tilde{r} := r / r^{\text{max}} \) and \( \tilde{\phi} = \phi / \phi^{\text{max}} \).
2. Solve with \( w_{\text{mp}} + w_{\text{obs}} = 1 \), \( w_{\text{mp}} \in \{10^{-2}, 10^{-1}, 0.5, 0.9, 0.99\} \).
3. Check values of the residuals \( r \) and \( \phi \).
Finding the optimal Pareto point in one shot

\[
\begin{align*}
\text{min}_{x, y, \theta} & \quad w_{mp} r(x, y, \theta) + w_{obs} \phi(Hx - z^{obs}) \\
\text{s.t.} & \quad Ax = b, \ x \geq 0 \\
 & \quad A^T y \leq F(x|\theta) \\
 & \quad \theta \in \Theta
\end{align*}
\]

Classic methodology to explore the Pareto curve

1. Normalize: \( \tilde{r} := r/r^{\text{max}} \) and \( \tilde{\phi} = \phi/\phi^{\text{max}} \).
2. Solve with \( w_{mp} + w_{obs} = 1 \), \( w_{mp} \in \{10^{-2}, 10^{-1}, 0.5, 0.9, 0.99\} \).
3. Check values of the residuals \( r \) and \( \phi \).

With noiseless data, sufficient to solve one program with \( w_{obs} \approx 1 \)

**Theorem 9 (Thai and Bayen 2014)**

If \( \exists x, y, \theta \) such that \( r(x, y, \theta) \leq \epsilon, \ Hx = z^{obs} \)

Then a solution \( (x^*, y^*, \theta^*) \) to the weighted sum method is such that

\[
\begin{align*}
& r(x^*, y^*, \theta^*) \leq \epsilon, \\
& \phi(Hx^* - z^{obs}) \to 0 \quad \text{as} \quad w_{obs} \to 1
\end{align*}
\]
Numerical experiments: weighted sum method

Theoretical results and implementation
Parallelization over multiple observations

Multiple observations:

- Given pairs \((z_{j}^{\text{obs}}, K_j)\) for \(j = 1, \cdots, N\)
- \(K_j = \{x \mid Ax = b_j, \; x \succeq 0\}\) encodes a specific configuration
- \(x_j\) is the resulting optimal response, but only observe \(z_{j}^{\text{obs}} = H x_j\)
- Find \(\theta\) and \(\{x_j\}_j\) solution to \(\text{VI}(K_j, F(\cdot|\theta))\), \(H x_j = z_{j}^{\text{obs}} \quad \forall j\)
Parallelization over multiple observations

- Given pairs \((z_{j}^{\text{obs}}, \mathcal{K}_j)\) for \(j = 1, \cdots, N\)
- \(\mathcal{K}_j = \{x \mid Ax = b_j, \ x \succeq 0\}\) encodes a specific configuration
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- Find \(\theta\) and \(\{x_j\}_j\) solution to \(\text{VI}(\mathcal{K}_j, F(\cdot | \theta))\), \(Hx_j = z_{j}^{\text{obs}} \forall j\)

\[
\begin{align*}
\min_{x,y,\theta} & \quad \ w_{\text{mp}} \ r(x, y, \theta) + w_{\text{obs}} \ \phi(Hx - z_{\text{obs}}) \\
\text{s.t.} & \quad Ax = b, \ x \succeq 0 \\
& \quad A^T y \preceq F(x | \theta) \\
& \quad \theta \in \Theta
\end{align*}
\]
Parallelization over multiple observations

- Given pairs \((z_{j}^{\text{obs}}, K_j)\) for \(j = 1, \cdots, N\)
- \(K_j = \{x \mid Ax = b_j, x \succeq 0\}\) encodes a specific configuration
- \(x_j\) is the resulting optimal response, but only observe \(z_{j}^{\text{obs}} = Hx_j\)
- Find \(\theta\) and \(\{x_j\}_j\) solution to \(\text{VI}(K_j, F(\cdot|\theta))\), \(Hx_j = z_{j}^{\text{obs}} \forall j\)

\[
\begin{align*}
\min_{x, y, \theta} & \quad w_{mp} \sum_j r(x_j, y_j, \theta) + w_{\text{obs}} \sum_j \phi(Hx_j - z_{j}^{\text{obs}}) \\
\text{s.t.} & \quad A_j x_j = b_j, \quad x_j \succeq 0, \quad j = 1, \cdots, N \\
& \quad A_j^T y_j \preceq F(x_j|\theta) \quad j = 1, \cdots, N \\
& \quad \theta \in \Theta
\end{align*}
\]
Parallelization over multiple observations

- Given pairs \((z_{j}^{\text{obs}}, K_{j})\) for \(j = 1, \cdots, N\)
- \(K_{j} = \{x \mid Ax = b_{j}, x \succeq 0\}\) encodes a specific configuration
- \(x_{j}\) is the resulting optimal response, but only observe \(z_{j}^{\text{obs}} = Hx_{j}\)
- Find \(\theta\) and \(\{x_{j}\}_{j}\) solution to VI\((K_{j}, F(\cdot|\theta))\), \(Hx_{j} = z_{j}^{\text{obs}} \forall j\)

\[
\begin{align*}
\min_{x, y, \theta} & \quad w_{\text{mp}} \sum_{j} r(x_{j}, y_{j}, \theta) + w_{\text{obs}} \sum_{j} \phi(Hx_{j} - z_{j}^{\text{obs}}) \\
\text{s.t.} & \quad A_{j}x_{j} = b_{j}, x_{j} \succeq 0, \quad j = 1, \cdots, N \\
& \quad A_{j}^{T}y_{j} \preceq F(x_{j}|\theta), \quad j = 1, \cdots, N \\
& \quad \theta \in \Theta
\end{align*}
\]

\(\theta\) is the common structural parameter. When fixed, parallelizable:

\[
\begin{align*}
\min_{x_{j}, y_{j}} & \quad w_{\text{mp}} r(x_{j}, y_{j}, \theta) + w_{\text{obs}} \phi(Hx_{j} - z_{j}^{\text{obs}}) \\
\text{s.t.} & \quad A_{j}x_{j} = b_{j}, x_{j} \succeq 0 \\
& \quad A_{j}^{T}y_{j} \preceq F(x_{j}|\theta) \quad \text{for} \ j = 1, \cdots, N
\end{align*}
\]
Convex and parallel block updates for solving the inverse problem

**New algorithm:** update cyclically blocks \( \{x_j\}_{j=1}^N \), \( \{y_j\}_{j=1}^N \), \( \theta \)

\[
\begin{align*}
\text{N Blocks } x_j & : \quad \min_{x_j} \, w_{mp} \, F(x_j|\theta)^T x_j + w_{obs} \|Hx_j - z_{j}^{\text{obs}}\|_2^2 \\
& \quad \text{s.t. } A_j x_j = b_j, \quad x_j \succeq 0, \quad A_j^T y_j \preceq F(x_j|\theta)
\end{align*}
\]

\[
\text{N Blocks } y_j : \quad \min_{y_j} \, -b_j^T y_j \quad \text{s.t. } A_j^T y_j \preceq F(x_j|\theta)
\]

Block \( \theta \) : \quad \min_{\theta} \, F(x|\theta)^T x \quad \text{s.t. } A^T y \preceq F(x|\theta), \quad \theta \in \Theta

In general, we have:

- affine parametrization in \( \theta \) in structural estimation
- nonnegative increasing convex cost function in economics
- hence convex in each block, except for constraint
- \( A_j^T y_j \preceq F(x_j|\theta) \)
- \( \Rightarrow \) also solves general bilevel programs with a sequence of convex programs.

Theoretical results and implementation 38
Convex and parallel block updates for solving the inverse problem

**New algorithm:** update cyclically blocks \( \{x_j\}_{j=1}^N, \{y_j\}_{j=1}^N, \theta \)

\[
\begin{align*}
N \text{ Blocks } x_j : \quad & \min_{x_j} \quad w_{mp} F(x_j|\theta)^T x_j + w_{obs} \|Hx_j - z_j^{obs}\|_2^2 \\
& \text{s.t. } A_j x_j = b_j, \ x_j \succeq 0, \ A_j^T y_j \preceq F(x_j|\theta) \\
\end{align*}
\]

\[
\begin{align*}
N \text{ Blocks } y_j : \quad & \min_{y_j} \quad -b_j^T y_j \quad \text{s.t. } A_j^T y_j \preceq F(x_j|\theta) \\
\end{align*}
\]

Block \( \theta : \quad & \min_{\theta} \quad F(x|\theta)^T x \quad \text{s.t. } A^T y \preceq F(x|\theta), \ \theta \in \Theta \\
\]

In general, we have:

- affine parametrization in \( \theta \) in structural estimation
- nonnegative increasing convex cost function in economics

hence convex in each block, except for constraint \( A_j^T y_j \preceq F(x_j|\theta) \)
Convex and parallel block updates for solving the inverse problem

**New algorithm:** update cyclically blocks \( \{x_j\}_{j=1}^N, \{y_j\}_{j=1}^N, \theta \)

\[
\begin{align*}
N \text{ Blocks } x_j : & \quad \min_{x_j} \quad w_{mp} F(x_j|\theta)^T x_j + w_{obs} \|H x_j - z_{j}^{\text{obs}}\|_2^2 \\
& \text{s.t. } A_j x_j = b_j, \ x_j \succeq 0, \quad A_j^T y_j \preceq F(x_j|\theta)
\end{align*}
\]

\[
\begin{align*}
N \text{ Blocks } y_j : & \quad \min_{y_j} \quad -b_j^T y_j \\
& \text{s.t. } A_j^T y_j \preceq F(x_j|\theta)
\end{align*}
\]

\[
\begin{align*}
\text{Block } \theta : & \quad \min_{\theta} F(x|\theta)^T x \\
& \text{s.t. } A^T y \preceq F(x|\theta), \ \theta \in \Theta
\end{align*}
\]

In general, we have:

- affine parametrization in \( \theta \) in structural estimation
- nonnegative increasing convex cost function in economics

hence convex in each block, except for constraint \( A_j^T y_j \preceq F(x_j|\theta) \)

\[ \implies \text{also solves general bilevel programs with a sequence of convex programs.} \]
Overall implementation: > 6000 lines of code in Python

Theoretical results and implementation
Implementation of the forward and reverse solvers

Forward solver

Graph delay functions

Conversion to CVXOPT Sparse matrices

A, b, F

CVXOPT

Eq. flow

Reverse solver

Graph

N demands

N observations

Conversion to CVXOPT Sparse matrices

A, \{b_j\}, \{z_j\}

θ update

θ_est

x_{j,est}

x_1, y_1 update

... 

x_j, y_j update

... 

x_N, y_N update

Theoretical results and implementation
Publications on previous projects

State Estimation for Polyhedral Hybrid Systems: Highway traffic estimation


State estimation for shallow water equations


Algorithmic contributions


Traffic density estimation from cellular network data

Publications based on the Forward-Inverse traffic assignment

https://github.com/jeromethai/traffic-estimation-wardrop

Submitted:


In preparation:


Summary of contributions since the Fall 2012

Inverse traffic assignment with missing data

- Formulation as a Pareto optimization program [7,8,10]
- Convex block updates for the Inverse and bilevel programs [8]
- Bounds on suboptimality gaps: $r_{\text{KKT}} \leq \epsilon \Rightarrow r_{\text{VI}} = O(\epsilon \|x\|_{\infty} n^{1-\frac{1}{p}})$ [7]
- Bounds on $x - x^*$: $r_{\text{KKT}} \leq \epsilon \Rightarrow \|x - x^*\| = O(\sqrt{\epsilon \|x\|_{\infty} n^{1-\frac{1}{p}} / m})$ [10]
- Optimal weights for the weighted sum method [7,10]
- Applications to the highway network of L.A. and consumer utility estimation [7,10]

Traffic estimation from cellular network data

- Fusion of cellular and loop data for route flow estimation [9]
- Link Density Inference from Cellular Infrastructure [6]

Hybrid estimation for highway traffic estimation

- The LWR Godunov with triangular flux is a switched linear systems [3]
- Hybrid estimation algorithms for highway traffic estimation [1,2]

Other contributions

- Inverse Covariance Estimation with missing data [5]

Theoretical results and implementation
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Theoretical results and implementation

Ongoing and future works
Sensitivity analysis of the inverse problem

\[ \| \hat{F} - F^{true} \| \leq \epsilon \implies \| \hat{x} - x^{true} \| \leq ? \]

where
\( \hat{x} \) is a solution to \( \text{VI}(K, \hat{F}) \)
\( x^{true} \) is a solution to \( \text{VI}(K, F^{true}) \)

What is the impact of estimate errors on control strategies?

- §5.6 of S. Boyd and L. Vandenberghe, *Convex Optimization*, 2004

Ongoing and future works
Cyber-security for the inverse problem

Attacks in noiseless case:

\[ z^{\text{obs}} = Hx + e \]

where \( Q = \text{supp}(e) \) = attacked entries

Mathematical program

\[
\begin{align*}
\min_{Q, \theta, x} & \quad |Q| \\
\text{s.t.} & \quad \text{supp}(Hx - z^{\text{obs}}) \subseteq Q \\
& \quad x \text{ is solution to } \text{VI}(K, F(\cdot|\theta))
\end{align*}
\]

Reformulation

\[
\min_{\theta, x} \|Hx - z^{\text{obs}}\|_0 \quad \text{s.t.} \quad x \text{ is solution to } \text{VI}(K, F(\cdot|\theta))
\]


Online learning and behavior fitting for the inverse problem

We fit an equilibrium model to measurements of traffic volumes to mimic more complex vehicle behaviors in an online, quasi-static setting.

Problem statement

- Slowly time-varying origin-destination demand
- Slowly time-varying traffic flow $x(t)$ (with complex traffic behavior)
- We only observe $z(t) = Hx(t)$
- Find $\theta(t)$ such that $\hat{x}(t)$ is sol. to $\text{VI}(\mathcal{K}, F(\cdot, \theta(t)))$ and fits $z(t)$

Usage

- Efficient rerouting in case of accidents
- Learning of vehicle behavior
Large-scale implementation of the inverse problem

<table>
<thead>
<tr>
<th>Model</th>
<th># links</th>
<th># nodes</th>
<th>forward pb.</th>
<th>reverse pb.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generic</td>
<td>$</td>
<td>\hat{A}</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>Highways</td>
<td>122</td>
<td>44</td>
<td>537</td>
<td>$730 \cdot N + d$</td>
</tr>
<tr>
<td>Hwy, main arterials</td>
<td>664</td>
<td>194</td>
<td>12,882</td>
<td>$16,645 \cdot N + d$</td>
</tr>
<tr>
<td>Hwy, all arterials</td>
<td>20,476</td>
<td>10,538</td>
<td>21.6M</td>
<td>$32.7M \cdot N + d$</td>
</tr>
</tbody>
</table>

Ongoing and future works
Integration of GPS, cellular and Twitter data

New vector objective

\[ [r(x, y, \theta), \phi^{\text{loop}}(H_1x - z_1), \phi^{\text{gps}}(H_2F(x) - z_2), \phi^{\text{cell}}(H_3 - z_3), \phi^{\text{tweet}}]^T \]

with  
\( z_1 = \) observed traffic volumes from loop detectors  
\( z_2 = \) observed delays from GPS data  
\( z_3 = \) observed traffic volumes from cellular network data

Figure: GPS traces and Twitter data generated by François Belletti.
Other applications

Cournot-Nash competition:


Controller fitting:

Thesis outline

1. Introduction
2. Estimation of the LWR Godunov for highway traffic estimation
   2.1. Highway traffic modeling
   2.2. LWR Godunov is a hybrid system
   2.3. Analysis of the space of modes
   2.4. Hybrid estimation for the LWR Godunov
   2.5. Numerical experiments on the I-880
3. Estimation of the road network in equilibrium from sparse data
   3.1. Wardrop equilibria and potential games
   3.2. Convex optimization, variational inequality, complementary problems
   3.3. Mathematical formulation of the inverse problem with missing data
   3.4. Bounds on suboptimality gap and distance to optimal solution
   3.5. Solution algorithm
   3.6. Large-scale implementation
4. Other applications for the inverse problem with missing data
   4.1. Consumer utility estimation and pricing
   4.2. Cournot-Nash competition
   4.3. Controlled fitting
5. Traffic estimation from other sources of data: GPS, Twitter, cellular network etc.
Appendix
Set of admissible functions \( \{ F(\cdot|\theta) \}_{\theta \in \Theta} \)

- **Parametric estimation** for \( \text{OP}(\mathcal{K}, f(\cdot|\theta)) \) and \( \text{VI}(\mathcal{K}, F(\cdot|\theta)) \)

\[
f(\cdot|\theta) = \sum_{i=1}^{d} \theta_i f_i,
F(\cdot|\theta) = \sum_{i=1}^{d} \theta_i F_i,
\theta \in \Theta \subseteq \mathbb{R}^d
\]
Set of admissible functions \( \{ F(\cdot|\theta) \} \theta \in \Theta \)

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- **Trivial solutions** exist, e.g. with \( \Theta = \mathbb{R}_+^d \) and \( \theta = 0 \), all \( \mathcal{K} \) is solution to

\[
\text{OP} : \min 0 \quad \text{s.t.} \quad x \in \mathcal{K} \quad \text{VI} : 0 = 0^T(u - x) \geq 0 \quad \forall u \in \mathcal{K}
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- **Normalization issue**: \( x \) sol to \( \text{OP}(\mathcal{K}, f) \) \( \Rightarrow \) \( x \) sol to \( \text{OP}(\mathcal{K}, \alpha f), \forall \alpha \geq 0 \)

**Appendix**
Set of admissible functions \( \{ F(\cdot|\theta) \} \) \( \theta \in \Theta \)

- **Parametric estimation** for \( \text{OP}(\mathcal{K}, f(\cdot|\theta)) \) and \( \text{VI}(\mathcal{K}, F(\cdot|\theta)) \)

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  \]

- **Normalization issue:** \( x \) sol to \( \text{OP}(\mathcal{K}, f) \) \( \Rightarrow \) \( x \) sol to \( \text{OP}(\mathcal{K}, \alpha f) \), \( \forall \alpha \geq 0 \)

- **A solution:** impose \( \alpha_1 = 1 \), hence \( f(\cdot|\theta) = f_1 + \sum_{i=2}^{d} \theta_i f_i \)
Set of admissible functions \( \{F(\cdot | \theta)\}_{\theta \in \Theta} \)

**Parametric estimation** for \( \text{OP}(\mathcal{K}, f(\cdot | \theta)) \) and \( \text{VI}(\mathcal{K}, F(\cdot | \theta)) \)

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**A solution**: impose \( \alpha_1 = 1 \), hence \( f(\cdot | \theta) = f_1 + \sum_{i=2}^{d} \theta_i f_i \)

**Prior information** in \( \Theta \), e.g., if the true \( f \) is convex then the following is sufficient for convexity of \( f(\cdot | \theta) \)

\[
\Theta = \mathbb{R}_+^d, \quad f_i \text{ convex} \quad \forall i = 1, \cdots, d
\]
Consumer behavior and pricing

company 3

consumer behavior?

demands
prices

guess on consumer behavior

pricing of product 3
Consumer behavior

Setting of the consumer behavior

- \( n \) products \( i = 1, \cdots, n \)

- **Demand** in product \( i \) is \( x_i \geq 0 \), hence overall demand \( x \in \mathbb{R}_+^n \)

- **Price** of product \( i \) is \( p_i \geq 0 \), hence overall pricing \( p \in \mathbb{R}_+^n \)

Consumer utility \( U(x) : \mathbb{R}_+^n \to \mathbb{R} \)

- Assumptions: \( U \) is **concave and increasing** on the range \([0, x_{\text{max}}]\)

- True utility function: \( U^{\text{true}}(x) = \mathbf{1}^T \sqrt{Ax + b} \)

- Parametric utility function: \( U(x \mid Q, r) = \frac{1}{2} x^T Q x + r^T x \)

- Prior information on the parametric utility function:

\[
\Theta = \{(Q, r) \mid Q x_{\text{max}} + r \geq 0, \ r \geq 0\}
\]

- Convex program:

\[
\begin{align*}
\max & \quad U(x) - p^T x \\
\text{s.t.} & \quad x \succeq 0
\end{align*}
\]
Estimation and pricing

- Parametric equilibrium model:
  \[
  (\mathbb{R}_+^n, F_j(\cdot|Q,r)), \quad j = 1, \cdots, N, \quad (Q, r) \in \Theta
  \]

- Payoff function:
  \[
  F_j(x_j|Q,r) = \nabla \{ p_j^T x - U(x_j|Q,r) \} = p_j^T - r - Qx_j
  \]

- Find optimal solution \((\tilde{Q}, \tilde{r})\) of:
  \[
  \min_{x,Q,r} \left[ \sum_j \|Hx_j - z_j\|^2, \sum_j r(x_j, Q, r) \right]^T
  \quad \text{s.t.} \quad (Q, r) \in \Theta
  \]

- Adjust price \(p_k\) given other prices:
  \[
  \tilde{f}(x|p_k) = p_kx_k + \sum_{i \neq k} p_ix_i - \tilde{U}(x)
  \]

- Adjust price \(p_k\) to achieve target demand \(z_k\) given other prices:
  \[
  \min_{x,p_k} \left[ (x_k - z_k)^2, r(x, p_k) \right]^T \quad \text{s.t.} \quad p_k \geq 0
  \]
Numerical example

- $n = 5$ products
- Sample $N = 200$ prices $p_j \in [p_{\text{min}}, p_{\text{max}}]$\(^5\)
- Compute induced demands $x_j$ for both models
- Missing data: demands for product 1, $p_{j1}$, $j = 1, \cdots, N$
Numerical example

2 models for $U^{\text{true}}(x) = \mathbf{1}^T \sqrt{\mathbf{A}x} + \mathbf{b}$

Appendix
Consumer utility estimation

Model 1, error = 26%

(a)

(b)

Model 2, error = 10%

(c)

(d)
Proof of theorem 5

\[ \exists y, \quad A^T y \leq F(x), \quad r_{PD} \leq \epsilon \]

\[ \iff \exists y, \quad A^T y \leq F(x), \quad F(x)^T x - b^T y \leq \epsilon \]

\[ \iff F(x)^T x - \max_y \{b^T y \mid A^T y \leq F(x)\} \leq \epsilon \]

\[ \iff F(x)^T x - \min_u \{F(x)^T u \mid u \in K\} \leq \epsilon \]

\[ \iff F(x)^T u \geq F(x)^T x - \epsilon, \quad \forall u \in K \]

\[ \iff r_{VI} \leq \epsilon \]
Proof of theorems 6 and 8

\[ \nu := F(x) - A^T y \quad \implies \quad \|[\alpha r_{\text{stat}}, r_{\text{comp}}]\|_p^p = \sum_i \alpha^p |\nu_i - \pi_i| + |x_i \pi_i|^p \]

Lemma

Let \( \alpha > 0, \ x \in \mathcal{K}, \ y \) such that \( A^T y \preceq F(x), \ p > 1 \). Then

\[ \min_{\pi \succeq 0} (r_{KKT}(x, y, \pi))^p = \sum_{i=1}^{n} \frac{(\nu_i x_i)^p}{(1 + (x_i / \alpha)^p)^{p-1}} \]

\[ \min_{\pi \succeq 0} r_{KKT} \geq \frac{\|\nu \circ x\|_p}{(1 + (\|x\|_{\infty}/\alpha)^{p-1})^{p-1}} \]

\[ \geq n^{1-p-1} \frac{r_{PD}}{(1 + (\|x\|_{\infty}/\alpha)^{p-1})^{p-1}} \]

We use: \( r_{PD}(x) \leq \epsilon \implies r_{VI}(x) \leq \epsilon \implies \|x - x^*\|_2 \leq \sqrt{\frac{\epsilon}{m}} \)
Estimation of discretized dynamic systems

\[ x_{n+1} = f(x_n, u_n) + w_n \]
\[ y_n = h(x_n) + v_n \]
Estimation of discretized dynamic systems

\[ x_{n+1} = f(x_n, u_n) + w_n \]
\[ y_n = h(x_n) + v_n \]

Extended Kalman filter (EKF)

**Prediction step**
\[ x_{n+1|n} = f(x_n, u_n, w_n) \]
\[ P_{n+1|n} = F_n P_n F_n^T + Q_n \]

**Update step**
\[ x_{n+1} = x_{n+1|n} + K_{n+1} r_{n+1} \]
\[ P_{n+1} = (I - K_{n+1} H_{n+1}) P_{n+1|n} \]
Estimation of discretized dynamic systems

\[ x_{n+1} = f(x_n, u_n) + w_n \]
\[ y_n = h(x_n) + v_n \]

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Linear approximation
\[ F_n = \partial_x f(x_n, u_n), \quad H_{n+1} = \partial_x h(x_{n+1|n}) \]
Estimation of discretized dynamic systems

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Linear approximation
\[ F_n = \partial_x f(x_n, u_n), \quad H_{n+1} = \partial_x h(x_{n+1|n}) \]

Cell Transmission Model (CTM) or discretized LWR PDE using Godunov
\[ f_i(x, 0) = x_i - \alpha \min(S(x_i), R(x_{i+1})) + \alpha \min(S(x_{i-1}), R(x_i)), \quad i \in [n] \]
Cell Transmission Model


Ensemble Kalman filter for highway traffic estimation (Mobile Millennium)

- Work et al., “An ensemble Kalman filtering approach to highway traffic estimation using GPS enabled mobile devices”, *CDC 2008*


Discretized highway model as a Hybrid system

Theorem 10 (Thai and Bayen 2013)

The discretized LWR PDE is an Autonomous Hybrid Automaton with affine components: given $x_n$, there exists $P_n$, $A_n$, $b_n$ such that

$$x_{n+1} = f(x_n, 0) = A_n x_n + b_n \quad \text{if} \quad x_n \in P_n$$

Theorem 11 (Thai and Bayen 2013)

There exists $N$ polyhedra $P_1, \cdots, P_N$ partitioning $[0, x_{max}]$ such that $N \sim 3 \cdot (2.25)^n$ as $n$ goes to $+\infty$ and $P_n$ is one of them $\forall x_n$.

▶ The condition $x_n \in P_n$ says that the Kalman filter (KF) applied to a linear component is in fact an EKF for the general system.

▶ There is an exponential number of polyhedra (modes) in $n$. Finding $P_n$ such that $x_n \in P_n$ and computing the associated update $A_n x_n + b_n$ can be done in linear time.
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The discretized LWR PDE is an Autonomous Hybrid Automaton with affine components: given $x_n$, there exists $P_n$, $A_n$, $b_n$ such that

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Theorem 11 (Thai and Bayen 2013)

There exists $N$ polyhedra $P_1, \cdots, P_N$ partitioning $[0, x_{\text{max}}]^n$ such that $N \sim 3.2 \cdot (2.25)^n$ as $n$ goes to $+\infty$ and $P_n$ is one of them $\forall x_n$. 
Discretized highway model as a Hybrid system

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if $x_n \in P_n$

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- The condition $x_n \in P_n$ says that the Kalman filter (KF) applied to a linear component is in fact an EKF for the general system.
- There is an \textbf{exponential number} of polyhedra (modes) in $n$.
- Finding $P_n$ such that $x_n \in P_n$ and computing the associated update $A_n x_n + b_n$ can be done in \textbf{linear time}. 
Efficient estimation algorithms

Appendix
Efficient estimation algorithms
Efficient estimation algorithms

Appendix 65

Hybrid estimation algorithm

Set $M$ of representative modes selected via clustering

Hybrid estimation algorithm

$\left( x_n^{(j)}, P_n^{(j)} \right)_{j \in M}$

Mixing step

$\left( x_n^{(j)}, P_n^{(j)} \right)_{j \in M}$

Kalman filter in each mode $j \in M$

$\left( x_{n+1}^{(j)}, P_{n+1}^{(j)} \right)_{j \in M}$

$n := n+1$
Fusion of cellular and loop data for route flow estimation

All flows are in 1000 vehicles/hour. 

Cellpath flows: 
\[
\begin{align*}
    f_{p1234} &= 1 = x_1 \\
    f_{p1654} &= 4 = x_2 \\
    f_{p654} &= 10 = x_3 + x_4
\end{align*}
\]

OD demands: 
\[
\begin{align*}
    d_{AB} &= 5 = x_1 + x_2 \\
    d_{CB} &= 10 = x_3 + x_4
\end{align*}
\]

Link flow: 
\[
b = 9 = x_2 + x_3
\]

Solving for the cellpath flow matrix: 
\[
(T \cdot x = d) : \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} d_{AB} \\ d_{CB} \end{bmatrix}
\]

Solving for the OD flow matrix: 
\[
(A \cdot x = b) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} f_{p1234} \\ f_{p1654} \\ f_{p654} \end{bmatrix}
\]

Solution with cellpaths: 
\[x^* = [1 \ 4 \ 5 \ 5]^T\]

Solution with ODs: 
\[x = x^* + [1 \ -1 \ 1 \ -1]^T t, \ \forall t \in [-1, 4]\]
Braess’s Paradox

Without road A-B

- Nash eq. $\iff$ same flow on ‘Start-A-End’ and ‘Start-B-End’
- Average delay $= 45 + \frac{2000}{100} = 65$ min

Examples
- Seoul, Stuttgart, NYC etc.
Braess’s Paradox

Without road A-B
- Nash eq. $\implies$ same flow on ‘Start-A-End’ and ‘Start-B-End’
- Average delay $= 45 + \frac{2000}{100} = 65$ min

Opens road A-B with very small delay
- Delay on ‘Start-A’ $\leq 4000/100 <$ delay on ‘Start-B’
- Nash eq. $\implies 4000$ along ‘Start-A-B-End’ $\implies$ Avg. delay $= 80$
Braess’s Paradox

Without road A-B
- Nash eq. $\iff$ same flow on ‘Start-A-End’ and ‘Start-B-End’
- Average delay $= 45 + \frac{2000}{100} = 65$ min

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- Nash eq. $\iff$ 4000 along ’Start-A-B-End’ $\iff$ Avg. delay $= 80$

Examples
- Seoul, Stuttgart, NYC etc.