1 Linear maps

Definition 1. Given a linear map $A : U \mapsto V$ then
the range space of $A$ is the subspace: $\mathcal{R}(A) := \{v \mid v = A(u), u \in U\} \subset V$
the nullspace of $A$ is the subspace: $\mathcal{N}(A) := \{u \mid A(u) = 0\} \subset U$

Note 1. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are linear subspaces.

Note 2. $A$ is injective $\iff \mathcal{N}(A) = \{0\}$ and $A$ is surjective $\iff \mathcal{R}(A) = V$

Proposition 1. Given a linear map $A : U \mapsto V$ with $\dim U = n$, then
$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n$$

Note 3. $\dim \mathcal{R}(A)$ is called the rank of $A$ and is denoted $\text{rk}(A)$.
$A$ is full rank $\Rightarrow \text{rk}(A) = \dim V \iff \mathcal{R}(A) = V$

Proposition 2. If $\dim U = \dim V = n$, then $A$ injective $\iff A$ surjective $\iff A$ bijective

Proof. Proof given as an exercise.

2 Orthogonal complement and projection

Definition 2. For a subspace $M \subset V$ (where $V$ is an inner product space), the orthogonal comple-
ment of $M$ is the set
$$M^\perp = \{y \in V : \langle x, y \rangle = 0 \ \forall x \in M\}$$

Problem 1. Verify: $M^\perp$ is a subspace of $V$ and $M \cap M^\perp = \{0\}$.

Definition 3. A projection is any linear map $P$ from a vector space to itself such that $P^2 = P$.

Definition 4. We define the orthogonal direct sum: $V = M^\perp \oplus M^\perp$:
$$\forall x \in V, \ \exists! x_1 \in M, \ \exists! x_2 \in M^\perp \text{ such that } x = x_1 + x_2$$
$$\forall x \in V, \ \exists! x_1 \in M \text{ such that } x - x_1 \in M^\perp$$

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Definition 5. From definition 4, we define the linear mappings $P_M(x) := x_1$ and $P_{M^\perp}(x) := x_2$.
We say that $P_M$ is the (orthogonal) projection onto $M$ and $P_M(x)$ is the projection of $x$ onto $M$.
Note that $P_M(x) + P_{M^\perp}(x) = x$.

Note 4. $(M^\perp)^\perp = M$ and $\dim M + \dim M^\perp = \dim V$.

Problem 2. Prove these properties:

a) $P_M(x) = x \iff x \in M \iff P_{M^\perp}(x) = 0$.
Equivalently, $\mathcal{R}(P_M) = M$, $\mathcal{N}(P_M) = M^\perp$, $\mathcal{R}(P_{M^\perp}) = M^\perp$, $\mathcal{N}(P_{M^\perp}) = M$.
b) $P_M^2 = P_M$
c) $\langle x, P_M(y) \rangle = \langle P_M(x), y \rangle = \langle P_M(x), P_M(y) \rangle \ \forall x, y$
3 Fundamental theorem of linear algebra

Definition 6. Let $U$ and $V$ be Hilbert spaces. Let $A : U \mapsto V$ be a linear map. Then the adjoint of $A$ is defined as the map $A^* : V \mapsto U$, such that:

$$\langle y, Ax \rangle_V = \langle A^* y, x \rangle_U \quad \forall x \in U, y \in V$$

Note 5. It follows from the definition of the adjoint that $(A^*)^* = A$.

Problem 3. Prove: a projection is orthogonal if and only if it is self-adjoint.

Proposition 3 (Fundamental theorem). $V = R(A) \oplus N(A^*)$ and $U = R(A^*) \oplus N(A)$

Equivalently, $R(A)^\perp = N(A^*)$, $N(A^*)^\perp = R(A)$, $R(A^*)^\perp = N(A)$, $N(A)^\perp = R(A^*)$

Proof. Proof given as an exercise.

Note 6. $A$ injective $\iff$ $A^*$ surjective and $A^*$ injective $\iff$ $A$ surjective.


Proof. Proof given as an exercise.

Problem 4. Show: $A$ surjective $\iff$ $AA^*$ bijective and $A$ is injective $\iff$ $A^*A$ bijective

Note 7. In practice, $U = \mathbb{R}^n$, $V = \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$, then

$$\forall x \in \mathbb{R}^n, \exists d \in \mathbb{R}^m, \exists r \in \mathbb{R}^n \quad \text{such that} \quad x = A^T d + r \quad \text{and} \quad Ar = 0$$

$A^T d$ is the (orthogonal) projection of $x$ onto $R(A^T)$ (i.e. $P_{R(A^T)}(x) = A^T d$).

$r$ is the projection of $x$ onto $N(A)$ (i.e. $P_{N(A)}(x) = r$).

In particular, we have $\langle A^T d, r \rangle = 0$, and $\|x\|_2^2 = \|A^T d\|_2^2 + \|r\|_2^2$

4 Applications

In the following problems, let $A$ be a real-valued matrix, i.e. $A \in \mathbb{R}^{m \times n}$.

Problem 5. Suppose $A$ is full rank (surjective) and non-injective. Let $x \in \mathbb{R}^n$:

a) Find closed-form expressions to $P_{R(A^T)}(x)$, $P_{N(A)}(x)$, $P_{R(A)}(x)$, and $P_{N(A^T)}(x)$.

b) Deduce expressions to $P_{R(A^T)}$, $P_{N(A)}$, $P_{R(A)}$, and $P_{N(A^T)}$, check they are orthogonal projections.

d) Answer a) and b) if $A$ is injective and non-surgeon.

Problem 6. Suppose $A$ is surjective and non-injective. Solve the following minimization problems:

$$\min_{x} \|x\|_2 \quad \text{s.t.} \quad Ax = b$$

$$\min_{x} c^T x \quad \text{s.t.} \quad Ax = b$$

Problem 7. Suppose $A$ is injective and non-surjective and $b \notin R(A)$. Solve the following minimization problem using the fundamental theorem of linear algebra:

$$\min_{x} \|Ax - b\|_2$$