1 MAP and MLE

Problem 1. For \( i \in \{0, 1\} \), when \( X = i, Y \sim \text{Poisson}(\mu_i) \). That is,
\[
P[Y = n | X = i] = \frac{\mu_i^n}{n!} e^{-\mu_i}, \quad n \geq 0
\]
The numbers \( \mu_i \) are known and such that \( \mu_0 < \mu_1 \). Assume that \( P(X = i) = p_i \), for \( i = 0, 1 \) where the numbers \( p_i \in (0, 1) \) are known and add up to one.
(a) Find MLE \( X | Y = n \).
(b) Find MAP \( X | Y = n \).

Solution. We calculate the likelihood ratio:
\[
L(n) = \frac{P[Y = n | X = 1]}{P[Y = n | X = 0]} = \left( \frac{\mu_1}{\mu_0} \right)^n e^{\mu_0 - \mu_1}
\]
(a) We know that MLE \( X | Y = n = 1 \) iff \( L(n) > 1 \), i.e. iff
\[
n(\log(\mu_1) - \log(\mu_0)) > \mu_1 - \mu_0 \iff n > \frac{\mu_1 - \mu_0}{\log(\mu_1) - \log(\mu_0)}
\]
(b) We know that MAP \( X | Y = n = 1 \) iff \( L(n) \frac{p_1}{p_0} > 1 \), i.e. iff
\[
n(\log(\mu_1) - \log(\mu_0)) > \mu_1 - \mu_0 + \log(p_0) - \log(p_1) \iff n > \frac{\mu_1 - \mu_0 + \log(p_0) - \log(p_1)}{\log(\mu_1) - \log(\mu_0)}
\]

Problem 2. Let \( X \) be uniformly distributed in \([0, 1]\). Assume that, given \( X = x \), the random variable \( Y \) is exponentially distributed with rate \( x + 1 \).
(a) Calculate \( E[Y] \).
(b) Find MLE \( X | Y = n \).
(c) Find MAP \( X | Y = n \).

Solution. (a) We first note that \( E[Y] = E[E[Y | X]] \). Computing \( E[Y | X] \):
\[
E[Y | X = x] = E[\exp(x + 1)] = 1/(x + 1)
\]
From above, \( E[Y] = E[E[Y | X = x]] = E[(1 + x)^{-1}] = \int_{x=0}^{1} (1 + x)^{-1} = [\ln(1 + x)]_0^1 = \ln 2 \)
(b, c) As \( X \) is uniformly distributed the MAP and MLE are equivalent. This means we just need to compute the MLE, i.e. the \( X \) that maximizes the likelihood of \( Y = y \):
\[
\text{MLE}[X | Y = n] = \arg \max_x f(Y = y | X = x)
= \arg \max_x (1 + x) e^{-(1+x)y}
\]
We can find this by calculating derivatives and setting it equal to zero:

\[
\frac{d}{dx} (1 + x)e^{-(x+1)y} = -y(1 + x)e^{-(x+1)y} + e^{-(x+1)y}y = e^{-(x+1)y}(-x - 1 + y) = 0
\]

\[
\implies x = \frac{1}{y} - 1
\]

Recognizing that \(x\) is also limited to be in the range of \([0, 1]\). Thus:

\[
\text{MAP}[X \mid Y = y] = \begin{cases} 
1 & \text{if } y \leq 0.5 \\
\frac{y}{2} - 1 & \text{if } 0.5 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

### 2 The Neyman Pearson Test

**Problem 3.** Let be \(n\) observations from a Gaussian distribution of unknown mean \(\mu\), but of known variance \(\sigma^2\). \(\mu\) can be either \(\mu_0\) (null hypothesis), or \(\mu_1\) (alternative hypothesis). Let’s assume \(\mu_1 > \mu_0\).

(a) Find the ratio of the likelihood

(b) What does the Neyman Pearson test tell us about the rejection of the hypothesis \(\mu = \mu_0\)?

(c) We know that \(P(x > 1.645) = 0.05\) when \(x \sim \mathcal{N}(0, 1)\). For \(\alpha = 0.05\), when is \(H_0\) rejected?

**Solution.**

(a) The likelihood of the observation is

\[
P(y \mid \mu) = \Pi_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_i - \mu)^2}{2\sigma^2}}
\]

\[
= \frac{1}{(\sqrt{2\pi})^n\sigma^n} \frac{e^{-\frac{(\sum_i y_i - \mu)^2}{2\sigma^2}}}{e^{-\frac{\sum_i y_i^2}{2\sigma^2}}} = \frac{1}{(\sqrt{2\pi})^n\sigma^n} e^{-\frac{(\mu - \sum_i y_i)^2}{2\sigma^2}}
\]

In the framework of the Neyman Pearson test, the ratio of the likelihood reads:

\[
\frac{P(y \mid \mu_1)}{P(y \mid \mu_0)} = Ke^{\frac{(\mu_1 - \mu_0)\sum_i y_i}{\sigma^2}}
\]

where \(K\) is a constant which does not depend on the observations.

(b) The Neyman Pearson theorem tells us to reject \(H_0\) if

\[
e^{\frac{(\mu_1 - \mu_0)\sum_i y_i}{\sigma^2}} > k
\]

where \(k\) is the generic term for a constant. Let define \(\bar{y} = \frac{\sum_i y_i}{n}\), i.e. the sample mean, hence \(\sum_i y_i = n\bar{y}\). Since \(\mu_1 - \mu_0 > 0\), the test will reject the hypothesis \(\mu = \mu_0\) if \(\bar{y} > \mu_c\), i.e. \(R = \{y \mid \bar{y} > \mu_c\}\) and \(A = \{y \mid \bar{y} \leq \mu_c\}\). The value of \(\mu_c\) is determined by the equation

\[
P(\bar{y} > \mu_c \mid H_0) = \alpha
\]

(c) The prior distribution on the observations is: \(\bar{y} \mid H_0 = \frac{1}{n} \sum_i y_i \sim \mathcal{N}(\mu_0, \sigma^2/n)\).

If we set \(\alpha = 0.05\), then \(P(\bar{y} > \mu_c \mid H_0) = \alpha\) corresponds to

\[
\mu_c = \mu_0 + 1.645\sigma/\sqrt{n}
\]