1 Markov chains

Problem 1. A mouse is in a maze and tries to escape. We assume that the mouse is a Markov mouse; i.e., the mouse moves randomly from room to room. Moreover, we assume that the mouse is equally likely to choose each of the available doors in the room it occupies (see Figure 1). There are two available doors in rooms 1, 2, and 3 and three available doors in room 4. Given that the mouse is initially in room 1, what is the average number of moves (or rooms visited) until the mouse escapes (in state 0)?

Solution. We model the movement of the mouse as a Markov chain. The state of the Markov chain is the room occupied by the mouse. We let the time index \( n \) refer to the \( n \)th room visited by the mouse. Specifically, we let \( X_n \) be the state (room) occupied by the mouse on step (or time or transition) \( n \). The initial room is \( X_0 = 1 \). The room after the first transition is \( X_1 \), and so forth. Then \( \{X_n\}_{n \geq 0} \) is a Markov chain. The mouse is in room \( X_n \) (a random variable) after making \( n \) moves, after having started in room \( X_0 = 1 \).

We proceed by first transition analysis. Let \( T_i = \min\{n \geq 1 : X_n = 0 | X_0 = i\} \), i.e. \( T_i \) is the number of moves until the mouse escapes given that it is initially in room \( i \). And we have the convention that \( T_0 = 0 \). We want to find \( E[T_1] \). We have:

\[
E[T_1] = E[T_1 | X_1 = 2]P(X_1 = 2 | X_0 = 1) + E[T_1 | X_1 = 3]P(X_1 = 3 | X_0 = 1) \\
= \frac{1}{2}E[T_1 | X_1 = 2] + \frac{1}{2}E[T_1 | X_1 = 3] \\
= \frac{1}{2}(1 + E[T_2]) + \frac{1}{2}(1 + E[T_3]) \\
= 1 + \frac{1}{2}(E[T_2] + E[T_3])
\]

Similarly,

\[
E[T_2] = 1 + \frac{1}{3}(E[T_1] + E[T_4]) \\
E[T_3] = 1 + \frac{1}{3}(E[T_1] + E[T_4]) \\
E[T_4] = 1 + \frac{1}{3}(E[T_2] + E[T_3])
\]

Solving these 4 linear equations gives \( E[T_1] = 13 \).

2 Binomial and Gaussian

Problem 2. Let \( W \sim \mathcal{N}(0, 1) \) and \( X \sim \mathcal{N}(\mu, \sigma^2) \). We recall that:

\[
P(W > 1.64) = 0.05 \\
P(W > 1.96) = 0.025 \\
P(W > 2.33) = 0.01
\]

Find \( a, b, c \) as a function of \( \mu \) and \( \sigma \) such that
Find the 90%, 95%, 98% confidence intervals for $X$ as a function of $\mu$ and $\sigma$.

**Solution.** Substituting $W = (X - \mu)/\sigma$ in $P(W > 1.64) = 0.05$, we have

$$P\left(\frac{(X - \mu)}{\sigma} > 1.64\right) = 0.05$$

in other words,

$$P\left(X > \mu + 1.64 \sigma\right) = 0.05$$

Similarly,

$$P\left(X > \mu + 1.96 \sigma\right) = 0.025$$
$$P\left(X > \mu + 2.33 \sigma\right) = 0.01$$

As derived in class,

$$P(-1.64 < W < 1.64) = 1 - P(W \leq -1.64) \cup P(W \geq 1.64)$$
$$= 1 - (P(W \leq -1.64) + P(W \geq 1.64))$$
$$= 1 - 2P(W \geq 1.64)$$
$$= 0.9$$

where the second equality holds because the events $\{W \leq -1.64\}$ and $\{W \geq 1.64\}$ are disjoint, and the third equality is from symmetry of the standard normal distribution.

Substituting $W = (X - \mu)/\sigma$:

$$P(-1.64 < (X - \mu)/\sigma < 1.64) = 0.9$$
$$P(\mu - 1.64 \sigma < X < \mu + 1.64 \sigma) = 0.9$$

hence $[\mu - 1.64 \sigma, \mu + 1.64 \sigma]$ is the 90% confidence interval for $X$. Similarly, $[\mu - 1.96 \sigma, \mu + 1.96 \sigma]$ and $[\mu - 2.33 \sigma, \mu + 2.33 \sigma]$ are the 95% and 98% confidence intervals for $X$ respectively.

**Problem 3.** You roll a dice 1000 times and count the number of times $K$ the face value is a factor of 3. What is the probability that $K$ is more than 358? What is the probability that $K$ is between 309 and 358? (Hint: we have $1.64 \times \sqrt{2000} \approx 74$)

**Solution.** We have $K \sim B(1000, 1/3)$ because there are 1000 trials and the probability of success is 1/3 (when the face value is 3 or 6). We know that $B(1000, 1/3) \approx N(1000/3, 2000/9)$. So $B(1000, 1/3)$ is close to a Gaussian distribution with mean $\mu = 1000/3$ and standard deviation $\sigma = \sqrt{2000}/9$. In particular, observing that

$$\mu + 1.64 \sigma = 1000/3 + 1.64 \times \sqrt{2000/9}$$
$$= (1000 + 1.64 \times \sqrt{2000})/3$$
$$\approx 1074/3$$
$$= 358$$

we conclude that

$$P(K \geq 358) \approx 0.05$$
Similarly, 
\[ \mu - 1.64\sigma = \frac{1000}{3} - 1.64 \times \sqrt{\frac{2000}{9}} \approx 309 \]
hence 
\[ P(309 \leq K \leq 358) \approx 0.9 \]

**Problem 4.** We are conducting a public opinion poll to determine the fraction \( p \) of people who will vote for Mr. Doe as the next president. 100 people took part in the poll and we found that 60\% of them answered in favor of Mr. Doe. Using this fact: \( p(1 - p) < \frac{1}{4}, \forall p \in [0, 1] \), find a rough estimate of the 95\% confidence interval for \( p \).

**Solution.** We model the number \( X \) of people in favor of Mr. Doe as a binomial random variable with parameters \( N = 100 \) and \( p \). We know that \( B(N, p) \approx N(Np, Np(1 - p)) \) hence 
\[ P(Np - 1.96\sqrt{Np(1 - p)} \leq X \leq Np + 1.96\sqrt{Np(1 - p)}) = 0.95 \]
hence 
\[ P \left( p - 1.96\sqrt{\frac{p(1 - p)}{N}} \leq \frac{X}{N} \leq p + 1.96\sqrt{\frac{p(1 - p)}{N}} \right) = 0.95 \]
in other words 
\[ P \left( \frac{X}{N} - 1.96\sqrt{\frac{p(1 - p)}{N}} \leq p \leq \frac{X}{N} + 1.96\sqrt{\frac{p(1 - p)}{N}} \right) = 0.95 \]
With \( N = 100 \), \( X/N = 0.6 \), we obtain 
\[ P \left( 0.6 - 1.96\sqrt{p(1 - p)} \leq p \leq 0.6 + 1.96\sqrt{p(1 - p)} \right) = 0.95 \]
hence \( [0.6 - 1.96\sqrt{p(1 - p)} \leq p \leq 0.6 + 1.96\sqrt{p(1 - p)}] \) is the 95\% confidence interval for \( p \).
Using, \( p(1 - p) < \frac{1}{4}, \forall p \in [0, 1] \), this confidence interval is contained in 
\[ [0.6 - 1.96\sqrt{1/4} \leq p \leq 0.6 + 1.96\sqrt{1/4}] = [0.502, 0.698] \]
so \([0.502, 0.698]\) is our estimate of the 95\% confidence