1 Basic probability

Problem 1 (independence). Let $A$, $B$, $C$ be three mutually independent events in some probability space. Show that $A \cup B$ and $C$ are independent.

Solution. We must show that

$$P((A \cup B) \cap C) = P((A \cup B)P(C)$$

Now,

$$P((A \cup B) \cap C) = P((A \cup B) \cap (B \cap C)) = P(A \cap B) + P(B \cap C) - P((A \cap B \cap C) = P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) = (P(A) + P(B) - P(A)P(B))P(C) = P(A \cup B)P(C)$$

Problem 2 (continuous r.v.). Let $X$, $Y$ be two independent $U[0,1]$ random variables. Calculate $P(X > Y^2)$.

Solution. We have

$$P(X > Y^2) = \int_0^1 P(X > y^2)P(Y \in (y, y + dy)) = \int_0^1 P(X > y^2)dy = \int_0^1 (1 - y^2)dy = 1 - \frac{1}{3} = \frac{2}{3}$$

Another way to see it

$$P(X > Y^2) = \mathbb{E}[(X > Y^2)] = \int_0^1 \int_0^1 \mathbb{1}(x > y^2)dx dy = \int_0^1 \int_y^1 1 dx dy = \int_0^1 (1 - y^2)dy = 1 - \frac{1}{3} = \frac{2}{3}$$

Problem 3 (Monty Hall paradox). This question is loosely based on the American television game show Let’s Make a Deal and named after its original host, Monty Hall.

Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, “Do you want to pick door No. 2?” Is it to your advantage to switch your choice?

What is the probability of winning by switching?
**Solution.** Assuming the player picks door 1, there are four possibilities. (i) The car is behind door 2 and the host opens door 3 with probability 1/3; (ii) the car is behind door 3 and the host opens door 2 with probability 1/3; (iii) The car is behind door 1 and the host opens door 3 with probability 1/6; (iv) The car is behind door 1 and the host opens door 2 with probability 1/6. For the first two possibilities, the player wins by switching, with probability 1/3+1/3=2/3, for the last two possibilities, the player loses by switching. Hence the probability of winning by switching is 2/3.

![Figure 1: Tree showing the probability of every possible outcome if the player initially picks Door 1.](image)

**Problem 4** (dice game). You have the option to throw a die up to three times. You will earn the face value of the die. You have the option to stop after each throw and walk away with the money earned. The earnings are not additive. What is the expected payoff of this game?

**Solution.** We proceed backwards.

Let’s suppose we have only 1 roll. Each roll is equally likely, so it will show 1,2,3,4,5,6 with equal probability. Thus their average of 3.5 is the expected payoff.

Now let’s suppose we have 2 rolls. If on the first roll, I roll a 6, I would not continue. The next throw would only maintain my winnings of 6 (with 1/6 chance) or make me lose. Similarly, if I threw a 5 or a 4 on the first roll, I would not continue, because my expected payoff on the last throw would be a 3.5. However, if I threw a 1,2 or 3, I would take that second round. This is again because I expect to win 3.5.

So in the 2 roll game, if I roll a 4,5,6, I keep those rolls, but if I throw a 1,2,3, I reroll. Thus I have a 1/2 chance of keeping a 4,5,6, or a 1/2 chance of rerolling. Rerolling has an expected return of 3.5. As the 4,5,6 are equally likely, rolling a 4,5 or 6 has expected return 5. Thus my expected payout on 2 rolls is 0.5(5)+0.5(3.5)=4.25.

Now we go to the 3 roll game. If I roll a 5 or 6, I keep my roll. But now, even a 4 is undesirable, because by rerolling, I’d be playing the 2 roll game, which has expected payout of 4.25. So now the expected payout is 0.166666... (4.25) = 4.66...

**Problem 5** (archer). The probability that an archer hits her target when it is windy is 0.4, when it is not windy her probability of hitting the target is 0.7. On any shot, the probability of a gust
of wind is 0.3. Find the probability that
(a) on a given shot there is a gust of wind and she hits her target.
(b) she hits the target with her first shot.
(c) there was no gust of wind on an occasion when she missed.

Solution. Let \( T \) be the event that she hits the target. Let \( W \) be the event that there is a gust of wind. We are told that \( P(T \mid W) = 0.4, P(T \mid W^c) = 0.7 \) and \( P(W) = 0.3 \).

(a) \( P(W \cap T) = P(T \mid W)P(W) = (0.4)(0.3) = 0.12 \)
(b) \( P(T) = P(T \mid W)P(W) + P(T \mid W^c)P(W^c) = (0.4)(0.3) + (0.7)(0.7) = 0.61 \)
(c) \( P(W^c \mid T^c) = \frac{P(T^c \mid W^c)P(W^c)}{P(T^c)} = \frac{(0.3)(0.7)}{(0.39)} = \frac{21}{13} = 0.538 \)

2 Markov chains

Problem 6. Consider the following transition probability matrices. Identify the transient and recurrent states, and the irreducible closed sets in each case. Justify.

(a) \[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0.2 & 0 & 0.8 & 0 \\
0.1 & 0.2 & 0.3 & 0.4 & 0 \\
0 & 0.6 & 0 & 0.4 & 0 \\
0.3 & 0 & 0 & 0 & 0.7 \\
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
\frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix}
\]

Solution. (a) There is one transient state: state 3. Indeed, starting at state 3 there is a positive probability that the Markov chain will never return to state 3.

The pair of states \( \{1, 5\} \) form a closed set (because if one starts from any one of these states the probability of leaving this set of states is zero) and they form an irreducible set of states (because they communicate with each other). Every finite closed set of states contains a recurrent state, and recurrence is a class property, so both the states 1 and 5 are recurrent. This can also be verified by checking that starting from state 1 the probability of returning to state 1 is one, and similarly, the probability of returning to state 5 when one starts at state 5 is one. The pair of states \( \{2, 4\} \) form a closed set (because if one starts from any one of these states the probability of leaving this set of states is zero) and they form an irreducible set of states (because they communicate with each other). By the same logic as in the preceding paragraph one can conclude that each of the states 2 and 4 is recurrent.

(b) Each of the states 3 and 6 is transient, because from either of states, starting in the state the Markov chain has a positive probability of never returning to that state.

By the same logic as in the preceding part of the problem, the pair of states \( \{1, 4\} \) form a closed set of states that is also irreducible. Further, by the same logic as in the preceding part of the problem
one can conclude that each of the states 1 and 4 is recurrent.
By the same logic as in the preceding part of the problem, the pair of states \{2, 5\} form a closed set of states that is also irreducible. Further, by the same logic as in the preceding part of the problem one can conclude that each of the states 2 and 5 is recurrent.