1 Continuous random variables

Problem 1. Let $X$ be uniformly distributed in $[0, 1]$. Assume that, given $X = x$, the random variable $Y$ is exponentially distributed with rate $x + 1$.
(a) Calculate $E[Y]$.
(b) Find $\text{MLE}[X | Y = n]$.
(c) Find $\text{MAP}[X | Y = n]$.

Problem 2. Let $X, Y$ be two independent $\text{Exp}(1)$ random variables. Calculate $E[X | X > y]$ (two different ways).

2 Markov chains

Problem 3. a) Give an example of a Markov chain $X_n$ on $\{0, 1, 2, 3\}$ and a function of the Markov chain that is not a Markov chain.
b) Give an example of a Markov chain $X_n$ on $\{0, 1, 2, 3\}$ and a function of that Markov chain that is not constant and not identical to $X_n$ and that is a Markov chain.

Problem 4. You roll a die until the sum of the last two rolls yields 9. What is the average number of rolls?

Problem 5. Consider the numbers 1, 2, · · · , 12 written around a ring as they usually are on a clock. Consider the Markov chain with state space $\{1, 2, \cdots, 12\}$ that at any time jumps with equal probability to one of the two adjacent numbers. What is the expected number of steps that the Markov chain will take to return to its original position?

3 Confidence intervals

Note 1. Let $X$ be a random variable with finite expected value $\mu$ and finite non-zero variance $\sigma^2$. Then for any real number $a > 0,$

$$P(|X - \mu| > a) \leq \frac{\sigma^2}{a^2}$$

Problem 6. In order to estimate the probability of head in a coin flip, $p$, you flip a coin $n$ times, and count the number of heads, $S_n$. You use the estimator $\hat{p} = S_n/n$. You choose the sample size $n$ to have a guarantee

$$P(|S_n/n - p| \geq \epsilon) \leq \delta$$

Determine how the value of $n$ suggested by Chebyshev inequality changes when $\epsilon$ is reduced to half of its original value? How does it change when $\delta$ is reduced to its original value?