1 Basic probability

Note 1. For events $A$, $B$, $C$ and in the same sample space, we have

- **Distribution:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Union:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Independence:** $A$ and $B$ independent iff $P(A \cap B) = P(A)P(B)$
- **Conditional probability:** $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(B \mid A) \frac{P(A)}{P(B)}$
- **Total probability:** $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A \mid B_n)P(B_n)$ with $\{B_n\}_n$ a partition of the sample space
- $\mathbb{E}[1_A] = P(A)$ with $1_A$ the indicator function of event $A$.

Note 2. Let $X$ and $Y$ be two continuous random variables

- **Expectation:** $\mathbb{E}[g(X,Y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x,y) f_{X,Y}(x,y) dx dy$
- **Total expectation:** $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$

Problem 1 (independence). Let $A$, $B$, $C$ be three mutually independent events in some probability space. Show that $A \cup B$ and $C$ are independent.

Problem 2 (continuous r.v.). Let $X$, $Y$ be two independent $U[0,1]$ random variables. Calculate $P(X > Y^2)$.

Problem 3 (Monty Hall paradox). This question is loosely based on the American television game show Let’s Make a Deal and named after its original host, Monty Hall.

Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, ”Do you want to pick door No. 2?” Is it to your advantage to switch your choice?

What is the conditional probability of winning by switching?

Problem 4 (dice game). You have the option to throw a die up to three times. You will earn the face value of the die. You have the option to stop after each throw and walk away with the money earned. The earnings are not additive. What is the expected payoff of this game?

Problem 5 (archer). The probability that an archer hits her target when it is windy is 0.4, when it is not windy her probability of hitting the target is 0.7. On any shot, the probability of a gust of wind is 0.3. Find the probability that

(a) on a given shot there is a gust of wind and she hits her target.
(b) she hits the target with her first shot.
(c) there was no gust of wind on an occasion when she missed.
2 Markov chains

Let \( \{X_n\}_{n} \) be a Markov chain on state space \( S = \{1, 2, 3, \cdots, N\} \).

Definition 1. The transition probability matrix \( P \) is defined such that \( P_{ij} = P(X_{n+1} = j \mid X_n = i) \). And note that \( P^k = P \times \cdots \times P \) has entries such that \( P^k_{ij} = P(X_{n+k} = j \mid X_n = i) \).

Definition 2. State \( j \) is accessible from \( i \) if there exists \( n \geq 0 \) and \( P^n_{ij} > 0 \). If \( i \) is accessible from \( j \) and \( j \) is accessible from \( i \), we say that \( i \) and \( j \) communicate (written \( i \leftrightarrow j \)).

Proposition 1. Communication classes (in which all states communicate) partition the state space. We say that the Markov chain is irreducible if it has only one class.

Definition 3. Let \( f_{ij} \) the probability that we ever reach state \( j \) from \( i \). We say that state \( i \) is recurrent if \( f_{ii} = 1 \), and transient if \( f_{ii} < 1 \).

Proposition 2. If we have \( i \leftrightarrow j \) and \( i \) is recurrent, then \( j \) is recurrent and \( f_{ij} = 1 \).

Definition 4. A set \( C \) is absorbing (or closed) if it is impossible to leave it, i.e. \( P_{ij} = 0 \) if \( i \in C \) and \( j \notin C \).

Problem 6. Consider the following transition probability matrices. Identify the transient and recurrent states, and the irreducible closed sets in each case. Justify.

(a)
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0.2 & 0 & 0.8 & 0 \\
0.1 & 0.2 & 0.3 & 0.4 & 0 \\
0 & 0.6 & 0.4 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0.7
\end{pmatrix}
\]

(b)
\[
\begin{pmatrix}
\frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0
\end{pmatrix}
\]