# Motion of Two Rigid Bodies with Rolling Constraint

ZEXIANG LI AND JOHN CANNY

*Abstract*—**Rolling constraint is a classical example of a nonholonomic constraint. Such a constraint is usually difficult to work with. In this paper, motion of two rigid bodies under rolling constraint is considered. In particular, the following two problems are being addressed: 1)** ***Given the geometry of the rigid bodies, determine the existence of an admissible path between two contact configurations.*** **2)** ***Assuming that an admissible path exists, find such a path.*** **First, the configuration space of contact is defined, the system of differential equations governing rolling constraint are derived. Then, a generalized version of the Frobenius's theorem, known as Chow's Theorem, to determine the existence of motion is applied. Finally, an algorithm is proposed that generates a desired path with one of the objects being flat. Potential applications of this study include 1) adjusting grasp configurations without slipping by a multifingered robot hand, 2) contour following without dissipation or wear by the end-effector of a manipulator, and 3) wheeled mobile robotics.**

## I. Introduction

Recently, there has been a great deal of interest in nonholonomic systems. For example, R. Brockett [3] studied the theory and control for a class of motors manufacutred by Panasonic Company [23]. Relying on the principle of *holonomy* (see [22]), this class of motors could excel, in terms of mass-to-torque ratio, the traditional dc motors by several orders of magnitude. T. Kane and M. Scher [16] looked at the falling cats problem. They explained how falling cats land on their feet even released from complete rest while upside-down; C. Frohlich [8] examined how a diver or a gymnast can do rotational maneuvers in midair without violating angular momentum conservation; M. Berry [1] studied the general shifting problem of a bead moving in a slowly rotating hoop. He established a general principle, known as the *holonomy principle*, underlying all the previous problems. J. Marsden, R. Montgomery, and R. Ratiu [12] presented a unified framework for systematically studying these problems.

In robotics research, recent effort has been focused on dexterous robot hands (see [17] and the references therein) which, due to rolling constraint and finger relocation, constitutes another example of nonholonomic systems. The well-known *dexterous manipulation* problem is to make use of the nonholonomic nature of the system so that the object can be manipulated from one grasp configuration to another.

Manuscript received January 3, 1989; revised July 19, 1989. This research was supported in part by the National Science Foundation under PYI Grant DMC 8451129 awarded to Prof. S. Sastry, in part by the David and Lucile Packard Foundation, and in part by the National Science Foundation under PYI Grant awarded to J. Canny. Part of the material in this paper was presented at the 5th International Symposium on Robotics Research, Tokyo, Japan, August 1989.

The authors are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720.

IEEE Log Number 8931959.

In this paper, we study motion of two rigid bodies under *rolling constraint*. This problem is a basic ingredient in dexterous manipulation. First, label the two rigid bodies by *obj1* and *obj2*, respectively (see Fig. 1). *Obj1* may represent the fingertip of a robot hand finger, and *obj2* the object being manipulated by the robot hand. This problem also has importance of its own. For example, in wheeled mobile robotics [19], *obj1* may represent the wheel (i.e., a ball wheel) of a mobile robot and *obj2* the curved surface where the robot travels. In contour following, *obj1* may represent the end-effector of a manipulator and *obj2* the workpiece.

By commanding rolling motion instead of sliding motion, which is known to be holonomic, the gained advantages are: 1) *The problem of wear associated with the contacting bodies is eliminated.* 2) *The associated control problem becomes much simpler.* Remember that in order to control sliding motion, the coefficient of friction has to be known exactly, which is in general difficult. Even the world's best figure skaters have trouble managing controlled sliding. On the other hand, rolling motion can be achieved by exerting forces which are sufficiently close to the center of the friction cone [6], [17]. 3) As we will see in this paper, *the set of configurations reachable by rolling is much larger than that reachable by sliding*. This is due to the nonholonomic nature of the constraint.

We address the following two problems in particular.

**Problem 1 (The Existence of Motion Problem)**: *Given two contact configurations, determine whether an admissible path exists between them.*

**Problem 2 (The Path Planning Problem)**: *Assuming that an admissible path exists (or a motion exists) between two contact configurations, find such a path.*

Motion planning with nonholonomic constraints is fundamentally different from motion planning with holonomic constraints. For the latter, a (semi-) algebraic description of the free space, in which a path can be planned, is available. The free space is specified either in terms of a set of equality, or inequality, constraints on the configuration variables [4] or in terms of a set of *integrable* differential equations (e.g., sliding). For the former, only a set of *nonintegrable* differential equations, which a path has to satisfy, is available.

An outline of the paper is as follows: In Section II, we review the geometries of a surface and the kinematics of contact. In Section III, we define the configuration space of contact and

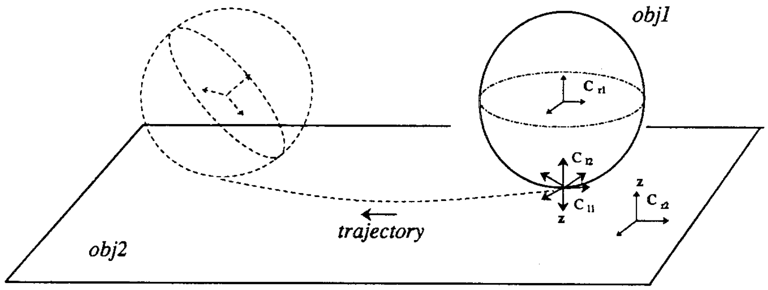


Fig. 1. Motion of an object with rolling constraints.

derive the system of differential equations that governs rolling motion. We then use some known results from differential geometry to determine the existence of a path. In Section IV, using geometric techniques, we present a simple algorithm that determines a desired path in the case when one of the objects is flat.

## II. Preliminaries

In this section, we review briefly the geometry of a surface and the kinematics of contact. See [14], [21], and [26] for further treatment on geometries of a surface and [7], [13], [17], [19], and [20] for the kinematics of contact.

**Notation 2.1**: Let $C_i$ and $C_j$ be two coordinate frames of $\mathbb{R}^3$, where $i$ and $j$ are arbitrary subscripts. Let $r_{i,j} \in \mathbb{R}^3$ and $R_{i,j} \in SO(3)$ denote the position and orientation of $C_i$ relative to $C_j$. The velocity of $C_i$ relative to $C_j$ is defined by

$$\begin{bmatrix} v_{i,j} \\ w_{i,j} \end{bmatrix} = \begin{bmatrix} R^t_{i,j}\dot{r}_{i,j} \\ \mathbb{S}^{-1}(R^t_{i,j}\dot{R}_{i,j}) \end{bmatrix}$$

where $\mathbb{S}$: $\mathbb{R} \to SO(3)$ identifies $\mathbb{R}^3$ with the space of $3 \times 3$ skew-symmetric matrices.

**Definition 2.1**: A space curve is the image of a $C^2$ map $c$: $I \to \mathbb{R}^3$, where $I$ is an interval. The pair $(c, I)$ is called a parameterization of the space curve. $c$ is regular if $\dot{c}(t) \neq 0$, $\forall\, t \in I$.

**Notation 2.2**: $U$ will always denote an open subset of $\mathbb{R}^2$. A point of $U$ will be denoted by $\boldsymbol{u} \in \mathbb{R}^2$, or by $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$, or $(u, v) \in \mathbb{R} \times \mathbb{R}$. Let $f$: $U \to \mathbb{R}^3$ be a differentiable map, $f_u$, $f_v$ denote the partial derivatives of $f$ with respect to $u$ and $v$, respectively.

**Definition 2.2**: A surface in $\mathbb{R}^3$ is a subset $S \subset \mathbb{R}^3$ such that for every point $s \in S$, there exists an open subset $S_s$ of $S$ with the property 1) $s \in S_s$, 2) $S_s$ is the image of a $C^3$ map $f$: $U \to \mathbb{R}^3$, where $f_u \times f_v \neq 0$, $\forall\, (u, v) \in U$, and 3) $f$: $U \to S_s \subset \mathbb{R}^3$ is a diffeomorphism.

$S_s$ is called a coordinate patch and the pair $(f, U)$ is called a (local) coordinate system of $S$. The coordinates of a point $s \in S_s$ are given by $(u, v) = f^{-1}(s)$. From now on, if the coordinate system is clear from the context, we shall not distinguish a point $s \in S_s$ from its coordinates. The collection of coordinate patches $\{S_s\}$ which covers $S$, i.e., $S = \cup S_s$, is called an atlas of $S$. By a curve in $S$ we mean a curve $c$: $I \to \mathbb{R}^3$, which can be expressed as $f \circ \boldsymbol{u}(t)$ for some curve $\boldsymbol{u}$: $I \to U$ in $U$.

**Example 2.1**: The sphere $S$ of radius $\rho$ is a surface. To see this, let $U = \{(u, v) \in \mathbb{R}^2, -\pi/2 < u < \pi/2, -\pi < v < \pi\}$ and consider the following coordinate systems:

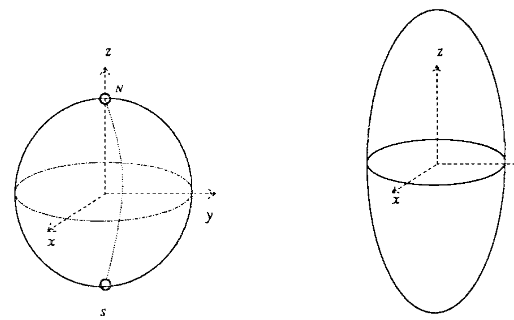


Fig. 2. (a) A sphere of radius $\rho$. (b) A football.

$$f\colon U \to \mathbb{R}^3$$
$$: (u, v) \mapsto (\rho \cos u \cos v, -\rho \cos u \sin v, \rho \sin u)$$

and

$$\hat{f}\colon U \to \mathbb{R}^3$$
$$: (u, v) \mapsto (-\rho \cos u \cos v, \rho \sin u, \rho \cos u \sin v).$$

The image of $f$ is the sphere minus the south pole, north pole, and an arc of the great circle connecting them (see Fig. 2(a)), i.e.,

$$f(U) = S - \{0, 0, \pm\rho\} \cup \{-\rho \cos u, 0, \rho \sin u\}, \quad -\pi/2 < u < \pi/2.$$

Similarly, the image of $\hat{f}$ is

$$\hat{f}(U) = S - \{0, \pm\rho, 0\} \cup \{\rho \cos u, \rho \sin u, 0\}, \quad -\pi/2 < u < \pi/2.$$

The partial derivatives of $f$ and $\hat{f}$ are

$$f_u = (-\rho \sin u \cos v, \rho \sin u \sin v, \rho \cos u)$$
$$f_v = (-\rho \cos u \sin v, -\rho \cos u \sin v, 0)$$

and

$$\hat{f}_u = (\rho \sin u \cos v, \rho \cos u, -\rho \sin u \sin v)$$
$$\hat{f}_v = (\rho \cos u \sin v, 0, \rho \cos u \cos v).$$

Clearly, $f_u \times f_v \neq 0$ and $\hat{f}_u \times \hat{f}_v \neq 0$, $\forall (u, v) \in U$. Moreover, $S_1 = f(U)$ and $S_2 = \hat{f}(U)$ cover $S$. Thus $S$ is a surface. ■

We denote by $S^2$ the unit sphere (i.e., $\rho = 1$) of $\mathbb{R}^3$.

**Example 2.2**: The football $x^2 + y^2 + (z^2/c^2) = 1$ (Fig. 2(b)) can be parametrized by the following coordinate system:

$$f: U \to \mathbb{R}^3: (u, v) \mapsto (\cos u \cos v, -\cos u \sin v, c \sin u)$$

and

$$\hat{f}: U \to \mathbb{R}^3: (u, v) \mapsto (-\cos u \cos v, \sin u, c \cos u \sin v)$$

where $U$ is given by the previous example. The reader may furnish the rest of the proof as an exercise. ■

**Definition 2.3**: The Gauss map of a surface $S$ is a continuous map $n: S \to S^2$ such that $n(s)$ is normal to $S$. We will also use $n$ to denote the map $n \circ f: U \to S^2$.

**Definition 2.4**: A coordinate system $(f, U)$ is called orthogonal if $f_u \cdot f_v = 0$, $\forall (u, v) \in U$, and right-handed if $f_u \times f_v / |f_u \times f_v| = n \circ f(\mathbf{u})$. Let $(f, U)$ be an orthogonal right-handed coordinate system for a surface patch $S_0 \subset S$. We define the Gaussian frame at a point $s \in S_0$ as the coordinate frame with origin at $f(\mathbf{u})$ and coordinate axes

$$x(\mathbf{u}) = f_u/|f_u| \qquad y(\mathbf{u}) = f_v/|f_v| \qquad \text{and} \qquad z(\mathbf{u}) = n \circ f(\mathbf{u}).$$

**Definition 2.5**: Let $S_0$ be a coordinate patch of $S$, with an orthogonal coordinate system $(f, U)$. At a point $s \in S_0$, the curvature form $K$ is defined as the $2 \times 2$ matrix

$$K = [x(\mathbf{u}), y(\mathbf{u})]^t [z_u(\mathbf{u})/|f_u|, z_v(\mathbf{u})/|f_v|]$$

where $\mathbf{u} = f^{-1}(s)$. The connection form $T$ is the $1 \times 2$ matrix

$$T = y(\mathbf{u})^t [x_u(\mathbf{u})/|f_u|, x_v(\mathbf{u})/|f_v|]$$

and the metric tensor $M$ is the $2 \times 2$ matrix

$$M = \begin{bmatrix} |f_u| & 0 \\ 0 & |f_v| \end{bmatrix}.$$

**Example 2.3**: Embed the plane in $\mathbb{R}^3$ by the following parameterization:

$$f: U \subset \mathbb{R}^2 \to \mathbb{R}^3: (u, v) \mapsto (u, v, 0).$$

The axes of the Gaussian frame are

$$x(\mathbf{u}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad y(\mathbf{u}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \quad z(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The curvature form, connection form, and metric tensor are

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad T = [0, \ 0] \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ ■

**Example 2.4**: Consider the sphere $S$ of radius $\rho$. Let $S_1 = f(U)$ be the coordinate patch of $S$ studied in Example 2.1. The Gaussian frame at a point $s \in S_1$ is given by

$$x(\mathbf{u}) = \begin{bmatrix} -\sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}$$

$$y(\mathbf{u}) = \begin{bmatrix} -\sin v \\ -\cos v \\ 0 \end{bmatrix}$$

and

$$z(\mathbf{u}) = \begin{bmatrix} \cos u \cos v \\ -\cos u \sin v \\ \sin u \end{bmatrix}.$$

The curvature form, connection form, and metric tensor are given by

$$K = \begin{bmatrix} 1/\rho & 0 \\ 0 & 1/\rho \end{bmatrix}$$

$$T = [0 \quad -\tan u/\rho]$$

and

$$M = \begin{bmatrix} \rho & 0 \\ 0 & \rho \cos u \end{bmatrix}.$$ ■

We now consider the two objects that move while maintaining contact with each other (see Fig. 1). Choose reference frames $C_{r1}$ and $C_{r2}$ fixed relative to $obj1$ and $obj2$, respectively. Let $S_1 \subset \mathbb{R}^3$ and $S_2 \subset \mathbb{R}^3$ be the embeddings of the surfaces of $obj1$ and $obj2$ relative to $C_{r1}$ and $C_{r2}$, respectively. Let $n_1$ and $n_2$ be the Gauss maps (outward normal) for $S_1$ and $S_2$. Choose atlases $\{S_{1,i}\}_{i=1}^{m_1}$ and $\{S_{2,i}\}_{i=1}^{m_2}$ for $S_1$ and $S_2$. Let $(f_{1,i}, U_{1,i})$ be an orthogonal right-handed coordinate system for $S_{1,i}$ with Gauss map $n_1$. Similarly, let $(f_{2,i}, U_{2,i})$ be an orthogonal, right-handed coordinate system for $S_{2,i}$ with $n_2$.

Let $c_1(t) \in S_1$ and $c_2(t) \in S_2$ be the positions at time $t$ of the point of contact relative to $C_{r1}$ and $C_{r2}$, respectively. We will restrict our attention to an interval $I$ such that $c_1(t) \in S_{1,i}$ and $c_2(t) \in S_{2,j}$ for all $t \in I$ and some $i$ and some $j$. The coordinate systems $(f_{1,i}, U_{1,i})$ and $(f_{2,j}, U_{2,j})$ induce a normalized Gaussian frame at all points in $S_{1,i}$ and $S_{2,j}$. We define a continuous family of coordinate frames, two for each $t \in I$, as follows. Let the local frames at time $t$, $C_{l1}$, and $C_{l2}$, be coordinate frames fixed relative to $C_{r1}$ and $C_{r2}$, respectively, that coincide at time $t$ with the normalized Gaussian frames at $c_1(t)$ and $c_2(t)$ (see Fig. 1).

We now define the parameters that describe the five degrees of freedom for the motion of the point of contact. The coordinates of the point of contact relative to the coordinate system $(f_{1,i}, U_{1,i})$ and $(f_{2,j}, U_{2,j})$ are given by $\mathbf{u}_1(t) = f_{1,i}^{-1}(c_1(t)) \in U_{1,i}$ and $\mathbf{u}_2(t) = f_{2,j}^{-1}(c_2(t)) \in U_{2,j}$.

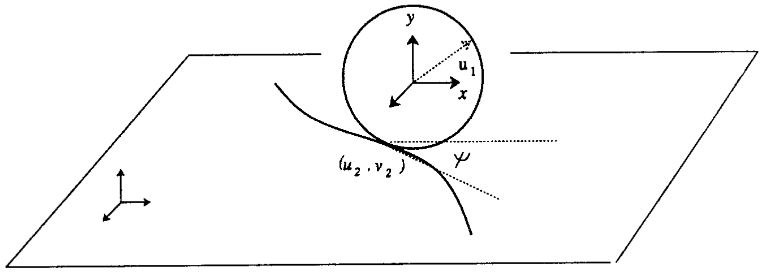



Fig. 3. A unit disc rolling over the plane.

These account for four degrees of freedom. The final parameter is the angle of contact $\psi(t)$, which is defined as the angle between the $x$ axis of $C_{l1}$ and $C_{l2}$. We choose the sign of $\psi$ so that a rotation of $C_{l1}$ through $-\psi$ around its $z$ axis aligns the $x$ axis.

We describe the motion of *obj1* relative to *obj2* at time $t$, using the local coordinate frames $C_{l1}$ and $C_{l2}$. Let $v_x$, $v_y$, and $v_z$ be the components of translation velocity of $C_{l1}$ relative to $C_{l1}$ at time $t$. Similarly, let $w_x$, $w_y$, and $w_z$ be the components of rotational velocity.

The symbols $K_1$, $T_1$, and $M_1$ represent, respectively, the curvature form, connection form, and metric tensor at time $t$ at the point $c_1(t)$ relative to the coordinate system $(f_{1,i}, U_{1,i})$. We can analogously define $K_2$, $T_2$, and $M_2$. We also let

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix} \qquad \tilde{K}_2 = R_\psi K_2 R_\psi.$$

Note that $\tilde{K}_2$ is the curvature of *obj2* at the point of contact relative to the $x$ and $y$ axes of $C_{l1}$. Call $K_1 + \tilde{K}_2$ the relative curvature form.

The following kinematic equations that describe motion of the point of contact over the surface of *obj1* and *obj2* in response to a relative motion between these objects are due to Montana [20].

**Theorem 2.1** (*Kinematic equations of contact*): *At a point of contact, if the relative curvature form is invertible, then the point of contact and angle of contact evolve according to*

$$\dot{u}_1 = M_1^{-1}(K_1 + \tilde{K}_2)^{-1}\left(\begin{bmatrix} -w_y \\ w_x \end{bmatrix} - \tilde{K}_2\begin{bmatrix} v_x \\ v_y \end{bmatrix}\right) \tag{1}$$

$$\dot{u}_2 = M_2^{-1}R_\psi(K_1 + \tilde{K}_2)^{-1}\left(\begin{bmatrix} -w_y \\ w_x \end{bmatrix} + K_1\begin{bmatrix} v_x \\ v_y \end{bmatrix}\right) \tag{2}$$

$$\dot{\psi} = w_z + T_1 M_1 \dot{u}_1 + T_2 M_2 \dot{u}_2 \tag{3}$$

$$0 = v_z. \tag{4}$$

The last equation is called the constraint equation.

**Example 2.5** (*The classical example revisited*): Let us consider the classical example of a unit disk rolling on the plane, as shown in Fig. 3 (see [9] and [10]). The coordinates of the plane are given by $(u_2, v_2) \in \mathbb{R}^2$, and the coordinate of the contact point on the disk is $u_1 \in \mathbb{R}$. Embed the disk into $\mathbb{R}^3$ with the following parametrization:

$$f\colon U_1 \subset \mathbb{R} \to \mathbb{R}^3\colon\ u_1 \mapsto (\cos u_1, \sin u_1, 0).$$

We define the Gaussian frame of the disc by the frame with origin at $f(u_1)$ and coordinate axes

$$x(u_1) = f' \qquad z(u_1) = f'' \qquad \text{and} \quad y(u_1) = z \times x.$$

Let $\psi$ be the angle of the disc relative to the $v_2$ axis. Let $(v_x, v_y, v_z)$ be the components of translational velocity of $C_{l1}$ relative to $C_{l2}$, and $(0, w_y, w_z)$ be the components of rotational velocity. *Note that the disc has only two degrees of rotational freedom.* Following a procedure outlined in [20], we derive the following kinematic equations of contact for the moving disc:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -1 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix} w_y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_z + \begin{bmatrix} 0 \\ \cos\psi \\ \sin\psi \\ 0 \end{bmatrix} v_x + \begin{bmatrix} 0 \\ -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} v_y$$

$$v_z = 0. \tag{5}$$

Rolling constraint implies that $(v_x = v_y = 0)$, and the above set of equations gets simplified to[1]

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -1 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix} w_y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_z \triangleq X_1 w_y + X_2 w_z. \tag{6}$$

$X_1$ and $X_2$ are called the "driving" and the "steering" vector fields, respectively. It is the direction of the corresponding infinitesimal motion.

■

[1] An alternative approach is to derive the constraint in differential forms, see [9], [10].

Rolling constraint is defined by the following conditions:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = 0 \quad \text{and} \quad w_z = 0. \tag{7}$$

Similarly, sliding constraint is defined by

$$\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = 0. \tag{8}$$

Substituting (7) into the kinematic equations of contact, yields

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} M_1^{-1} \\ M_2^{-1} \\ T_1 + T_2\tilde{R}_\phi \end{bmatrix} (K_1 + \tilde{K}_2)^{-1} \begin{bmatrix} -w_y \\ w_x \end{bmatrix}. \tag{9}$$

## III. Existence of Motion

In this section, we use the kinematic equations of contact and a generalized version of the Frobenius Theorem to determine the existence of an admissible path between two contact configurations.

**Definition 3.1:**[2] The configuration space of contact $P$ is a five-dimensional space, which locally is described by the coordinates of contact relative to *obj1* and *obj2*, and the angle of contact, i.e., a contact configuration $p \in P$ has the form

$$p = (u_1, v_1, u_2, v_2, \phi)^T.$$

Note that this definition of $P$ depends on the coordinate systems used for *obj1* and *obj2*. An intrinsic definition of $P$ is given in [18].[3]

Consider now the kinematic equations of contact with rolling constraint imposed, which can be rewritten in the form

$$\dot{p} = X_1(p)w_x + X_2(p)w_y \qquad \dot{p} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\phi} \end{bmatrix} \tag{10}$$

where

$$X_1(p) = \begin{bmatrix} M_1^{-1} \\ M_2^{-1} \\ T_1 + T_2\tilde{R}_\phi \end{bmatrix} (K_1 + \tilde{K}_2)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X_2(p) = \begin{bmatrix} M_1^{-1} \\ M_2^{-1} \\ T_1 + T_2\tilde{R}_\phi \end{bmatrix} (K_1 + \tilde{K}_2)^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{11}$$

[2] We assume that the relative curvature form is invertible.

[3] For readers familiar with differential geometry, $P$ is defined as follows: Let $T_0S_1$ be the circle bundle of $S_1$ and $T_0S_2$ the circle bundle of $S_2$. Form the product space $(T_0S_1 \times T_0S_2)$ and let $S^1$, the circle group, acting on $T_0S_1$ by left rotation and on $T_0S_2$ by right rotation (i.e., we have a diagonal action of $S^1$ on $(T_0S_1 \times T_0S_2)$). Then $P$ is the product space quotient the diagonal action, i.e., $P = (T_0S_1 \times T_0S_2)/S^1$ (see [18] and [28]).

Equation (10) defines a system of differential equations on $P$. $X_1(p)$ and $X_2(p)$ are the vector fields for the infinitesimal rolling motion.

**Definition 3.2**: A path $p(t) \in P$, $t \in [0, \infty)$, is said to be admissible (or conforms with the constraint) if it satisfies the differential equation (10) for some piecewise-continuous rolling velocity $(w_x(t), w_y(t)) \in \mathbb{R}^2$, $t \in [0, \infty)$.

**Definition 3.3**: Let $p_0 \in P$ be an initial contact configuration. A point $p_f \in P$ is said to be reachable from $p_0$ by rolling if there exists an admissible path $p(t) \in P$, $t \in [0, t_f]$, such that $p(0) = p_0$ and $p(t_f) = p_f$.

The following is a restatement of the existence of motion problem.

**Problem 1′ (The Existence of Motion Problem)**: *Given two contact configurations $p_0, p_f \in P$, determine the existence of an admissible path that connects $p_0$ to $p_f$.*

Modifying a result from differential geometry, known as the Chow's Theorem [5], we arrive at the following algorithm that solves Problem 1′. A proof of correctness of the algorithm can be found in [11] and [24].

**Algorithm 3.1** (*Existence of Motion Algorithm*)

**Input:** 1) Coordinate systems $\{f_{1,i}, U_{1,i}\}_{i=1}^{i=m_1}$ of *obj1*, and $\{f_{2,i}, U_{2,j}\}_{j=1}^{j=m_2}$ of *obj2*.
2) Geometrical data $(M_1, T_1, K_1)$ of *obj1* and $(M_2, T_2, K_2)$ of *obj2*.
3) The coordinates of two contact configurations $p_0, p_f \in P$.

**Output:** Determine if $p_f$ can be reached from $p_0$ by rolling.

**Step 1:** Compute the coordinate expressions of the vector fields $X_1(p)$ and $X_2(p)$ from (11).

**Step 2:** Compute the following Lie bracket vector fields (see the remark that follows)

$$\begin{aligned} X_3(p) &= [X_1, X_2] = \frac{\partial X_2}{\partial p_i}X_1 - \frac{\partial X_1}{\partial p_i}X_2 \\ X_4(p) &= [X_1, X_3] \\ X_5(p) &= [X_2, X_3] \end{aligned} \tag{12}$$

where $p = (u_1, v_1, u_2, v_2, \psi)^t$.

**Step 3:** Form the distribution[4]

$$\nabla(p) = \{X_1, X_2, X_3, X_4, X_5\}. \tag{13}$$

For each $p \in P$, $\nabla(p)$ is a $5 \times 5$ matrix. Compute the rank of $\nabla(p)$.

**Output:** a) If rank $(\nabla(p)) = 5$, $\forall p \in P$, then there exists an admissible path between any two contact configurations.[5]
b) If $\dim(\nabla(p)) = n < 5$, $\forall p \in P$,[6] let $N_{p_0}$ be the maximum integral manifold of $\nabla$

[4] For each $p \in P$, $\nabla(p)$ is an involutive distribution, known as the Lie algebra generated by $\{X_1(p), X_2(p)\}$.

[5] This says that if $\nabla(p)$ is full rank, then any point in the space can be reached by moving along the integral curves of $X_1$ and $X_2$.

[6] For technical reasons we assume that $\nabla(p)$ has constant rank. Otherwise see [11], [24].

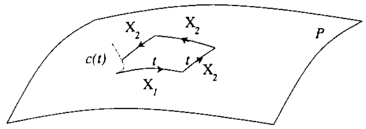



Fig. 4. An interpretation of $[X_1, X_2]$.

through $p_0$.[7] If $p_f \in N_{p_0}$, then an admissible path exists between $p_0$ and $p_f$.[8]

c) Otherwise, no path exists.

**Remark 3.1**: 1) The Lie bracket vector field has the following meanings: Let $X_1$ and $X_2$ be two vector fields on $P$, and $p \in P$. Define a curve $c$ on $P$ as follows. For sufficiently small $t$, 1) follow the integral curve of $X_1$ through $p$ for time $t$; 2) starting from there, follow the integral curve of $X_2$ for time $t$; 3) then follow the integral curve of $X_1$ backwards for time $t$; 4) then follow the integral curve of $X_2$ backwards for time $t$ (see Fig. 4). In other words

$$c(t) = \Psi_{-t}(\Phi_{-t}(\Psi_t(\Phi_t(p))))$$

where $\Phi_t$, $\Psi_t$ are the integral curves of $X_1$ and $X_2$, respectively. Then, we have

$$\ddot{c}(0) = 2[X_1, X_2](p).$$

2) The previous remark also suggests a way of creating a net motion in the direction $[X_1, X_2]$ by moving along the directions $X_1$ and $X_2$.

3) Computation of the Lie bracket vector fields, and checking the rank of $\nabla(p)$ can be done using Macsyma.

We now apply the above algorithm to several examples.

**Example 3.1**: Consider a unit ball rolling on the plane, as shown in Fig. 1. From Examples 2.1 and 2.3, the ball has two coordinate systems, and the plane one. The curvature forms, metric tensors, and connection forms are given in Example 2.4 and 2.3, respectively.

**Step 1**: On the first coordinate system of $P$, the kinematic equations of contact are

$$\begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ \sec u_1 \\ -\sin\psi \\ -\cos\psi \\ -\tan u_1 \end{bmatrix} w_x + \begin{bmatrix} -1 \\ 0 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix} w_y$$
$$\triangleq X_1(p) w_x + X_2(p) w_y. \tag{14}$$

**Step 2**: Computing the successive Lie brackets of $X_1(p)$ and $X_2(p)$, gives

$$X_3 = [X_1, X_2] = \begin{bmatrix} 0 \\ -\sec u_1 \tan u_1 \\ -\sin\psi \tan u_1 \\ \cos u_1 \tan u_1 \\ -\sec^2 u_1 \end{bmatrix}$$

$$X_4 = [X_1, X_3] = \begin{bmatrix} 0 \\ 0 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix}$$

and

$$X_5 = [X_2, X_3] = \begin{bmatrix} 0 \\ (1+\sin^2 u_1)\sec^3 u_1 \\ 2\sin\psi \sec^2 u_1 \\ 2\cos\psi \sec^2 u_1 \\ 2\sec^2 u_1 \tan u_1 \end{bmatrix}.$$

**Step 3**: Form the distribution

$$\nabla = \{X_1, X_2, X_3, X_4, X_5\}.$$

It is easy to verify that, through elementary row and column operations, the determinant of $\nabla$ is identically 1.

Steps 1) through 3) are repeated for the second coordinate system of $P$ and $\nabla$ is again nonsingular.

**Output:** *It is true that a unit ball can reach any contact configuration on the plane by rolling*!

■

**Example 3.2**: The second example consists of a unit ball rolling on another ball of radius $\rho$ (see Fig. 5). By the previous example, $P$ has four coordinate systems.

**Step 1**: The kinetic equations of contact in the first coordinate system are

$$\begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ (1-\beta)\sec u_1 \\ -\beta\sin\psi \\ -\beta\cos\psi\sec u_2 \\ \beta\tan u_2\cos\psi - (1-\beta)\tan u_1 \end{bmatrix} w_x + \begin{bmatrix} -(1-\beta) \\ 0 \\ -\beta\cos\psi \\ \beta\sin\psi\sec u_2 \\ -\beta\tan u_2\sin\psi \end{bmatrix} w_y$$
$$\triangleq X_1 w_x + X_2 w_y$$

where $\beta = 1/(1+\rho)$.

**Step 2**: Using Macsyma, the successive Lie brackets of $X_1$ and $X_2$ are computed.

$$X_3 = [X_1, X_2] = \begin{bmatrix} 0 \\ (1-\beta)^2\sec^2 u_1 \\ \beta(1-\beta)\sin\psi\sin u_1\sec u_1 \\ \beta(1-\beta)\sin\psi\sin u_1\sec u_1\sec u_2 \\ X_{3,5} \end{bmatrix}$$

[7]The existence and uniqueness of $N_{p_0}$ is guaranteed by Frobenius Theorem.
[8]This condition is rather difficult to check, see [25].

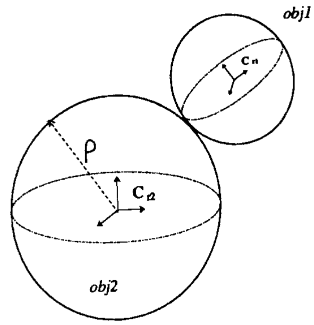



Fig. 5. Motion of a unit ball over another ball.

where

$$X_{3,5} = -\frac{\beta(1-\beta)\cos\psi\cos u_1 \sin u_1 \sin u_2 + \{-\beta^2\cos^2 u_1 + (\beta-1)^2\}\cos u_2}{\cos^2 u_1 \cos^2 u_2}$$

$$X_4 = [X_1, X_3] = \begin{bmatrix} 0 \\ 0 \\ \beta(2\beta-1)\cos\psi \\ -\beta(2\beta-1)\sin\psi\sin u_2 \sec u_2 \\ \beta(2\beta-1)\sin\psi\sin u_2\sec u_2 \end{bmatrix}$$

$$X_5 = [X_2, X_3] = \begin{bmatrix} 0 \\ -\{-(1-\beta)^3\cos^2 u_1 + 2(1-\beta)^3\}\sec^3 u_1 \\ -\{\beta^3 \sin\psi\cos^2 u_1 - 2\beta(1-\beta)^2\sin\psi\}\sec^2 u_1 \\ -\{\beta^3\cos\psi\cos^2 u_1 - 2\beta(1-\beta)^2\cos\psi\}\sec^2 u_1 \sec u_2 \\ X_{5,5} \end{bmatrix}$$

where

$$X_{5,5} = \frac{\{\beta^3\cos\psi\cos^3 u_1 - 2\beta(1-\beta)^2\cos\psi\cos u_1\}\sin u_2 + \alpha}{\cos^3 u_1 \cos u_2}$$

and

$$\alpha = \{\beta^2(1-\beta)\cos^2 u_1 - 2(1-\beta)^3\}\sin u_1 \cos u_2.$$

**Step 3**: Computing the determinant of

$$\nabla = \{X_1, X_2, X_3, X_4, X_5\}$$

gives

$$\det \nabla = -\frac{(\beta-1)^2\beta^2(2\beta-1)^3}{\cos u_1 \cos u_2}, \qquad \beta = \frac{1}{1+\rho}.$$

$\nabla$ is singular for the following cases:

- $\beta = 1 \rightarrow \rho = 0$: This corresponds to *obj2* being a single point. Note that the rank of $\nabla$ is 3 (not 2!). This can also be seen from the multiplicity of the zeros in the determinant.
- $\beta = \frac{1}{2} \rightarrow \rho = 1$: This corresponds to the case when both objects are balls of identical radius. In fact, counting the multiplicity of the zeros at $\beta = \frac{1}{2}$, or computing the rank of $\nabla$, the reachable space has dimension 2! This fact can be interpreted using the notion of holonomy angles (see Section IV).
- $\beta = 0 \rightarrow \rho = \infty$. The result is degenerate because from the previous example we know that a unit ball can reach any contact configuration on the plane by rolling.

Steps 1) through 3) are repeated for the other three coordinate systems and the results are consistent.

**Output**: *It is true that a unit ball can reach any contact configuration by rolling on another ball of radius $\rho$ if and only if $\rho$ is not zero or ($\rho \neq 1$).*

■

**Example 3.3** (*The classical example revisited*): Consider again the classic example of a unit disc on the plane. Note that the two rotations are different here from Example 3.1. We get from Example 2.5 the following two vector fields:

$$\text{“driving”} = X_1 = \begin{bmatrix} -1 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix}$$

and

$$\text{“steering”} = X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Performing the Lie bracket operation, gives

$$X_3 = [X_1, X_2] = \begin{bmatrix} 0 \\ -\sin\psi \\ -\cos\psi \\ 0 \end{bmatrix}$$

and

$$X_4 = [X_2, X_3] = \begin{bmatrix} 0 \\ -\cos\psi \\ \sin\psi \\ 0 \end{bmatrix}.$$

Note that $[X_1, X_3] = 0$. $X_3$ and $X_4$ are called the "wriggling" and the "sliding" vector fields, respectively. It is then simple to verify that

$$\nabla = \{X_1, X_2, X_3, X_4\}$$

has rank 4, for all points in $P$. This shows that a disk can reach any contact configuration by "driving" and "steering." ■

## IV. A Path Planning Algorithm

This section is devoted to the solution of the following planning problem.

**Problem 2′ (Path Planning Problem)**: *Assuming that an admissible path exists between two contact configurations $p_0, p_f \in P$, find one path.*

One approach is to consider it as a *nonlinear control problem*. The *plant equation* is given by (10), whereas $p(t) \in P$ is the state, $(X_1(p), X_2(p))$ are the control vector fields, and $(w_x, w_y) \in \mathbb{R}^2$ the control inputs. The objective is to find a set of control inputs $(w_x(t), w_y(t)) \in \mathbb{R}^2$, $t \in [0, t_f]$, such that the system (10), starting from $p_0$, reaches $p_f$ in finite time. Relevant works in nonlinear control literature include [3], [11], [24].

Making use of the contact constraint, an alternative approach is presented here. First, from our *driving* experiences, we know that a path relative to the surface of *obj1* (or *obj2*) determines uniquely a path in the configuration space of contact. More precisely, we have

**Proposition 4.1**: *Let $p_0 = \{u_1(0), u_2(0), \psi(0)\} \in P$ be an initial contact configuration. Then, a path $u(t) \in S_1, t \in [0, t_f]$, determines uniquely a path $p(t) \in P$, $t \in [0, t_f]$.*[9]

**Proof**: It suffices to show that $(u_2(t), \psi(t))$ are uniquely determined by $u_1(t)$, $t \in [0, t_f]$. But, from (1), rolling velocity can be expressed in terms of $\dot{u}_1$ as

$$\begin{bmatrix} -w_y \\ w_x \end{bmatrix} = (K_1 + \tilde{K}_2) M_1 \dot{u}_1. \quad (15)$$

[9]When the coordinate system in consideration is clear, we shall not distinguish the object surface from its coordinates in order to simplify notation.

Substituting this into (2) and (3) yields

$$\begin{bmatrix} \dot{u}_2 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} M_2^{-1}\tilde{R}_\psi \\ T_1 + T_2\tilde{R}_\psi \end{bmatrix} M_1 \dot{u}_1. \quad (16)$$

For given initial conditions $(u_2(0), \psi(0))$, a theorem (the existence and uniqueness theorem) of ODE ensures the existence and uniqueness of the solution to (16). This completes the proof. □

We call the solution, $p(t) = (u_1(t), u_2(t), \psi(t))$, $t \in [0, t_f]$, from (16) the *lift* of the path $u_1(t)$ through the point $p_0$. Apparently, the lift $p(t) \in P$ is admissible, or satisfies the rolling constraint.

**Corollary 4.1**: *Let $p_0 \in P$ be an initial contact configuration and $u_2(t) \in S_2$, $t \in [0, t_f]$, a contact trajectory relative to obj2. Then, there exists a unique lift $p(t) \in P$, $t \in [0, t_f]$, defined by the following ODE*:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} M_1^{-1}\tilde{R}_\psi \\ T_1\tilde{R}_\psi + T_2 \end{bmatrix} M_2 \dot{u}_2. \quad (17)$$

The angle of contact $\psi$, whose evolution is defined by (16), has a useful geometric interpretation when *obj2* is flat, i.e., $T_2 = 0$. Let $u_1(t)$, $t \in [t_0, t_1]$, be a piecewise-regular simple closed curve in $S_1$ representing the contact trajectory of *obj1*, and $\delta\psi = \psi(t_1) - \psi(t_0)$ denote the net change of contact angle induced by $u_1$. We have

**Proposition 4.2**: *$-\delta\psi$ is equal to the holonomy angle of the loop $u_1$ (see [27] for the definition of holonomy angle). In other words, $-\delta\psi = \iint_R k\, dA$, where $k$ is the Gaussian curvature of $S_1$ and $R$ is the region bounded by $u_1$.*

**Remark 4.1:** This is a key result to the path finding algorithm. In order to realize a desired change of contact angle without altering the point of contact relative to $S_1$, we may plan a closed curve in $S_1$ such that the Gaussian curvature integral over the region bounded by the loop is equal to the net angle change.

**Proof:** This follows from the Gauss–Bonnet Theorem in differential geometry. For details see [14], [18], [27]. □

Using (17), (16), and Proposition 4.2, we have the following algorithm that generates a desired path when *obj2* is flat. The example of a unit ball on the plane is used for illustration.

**Algorithm 4.1** (*A Path Finding Algorithm*)

**Input:** 1) Initial and final configurations $p_0 = (u_1^0, u_2^0, \psi^0)$ and $p_f = (u_1^f, u_2^f, \psi^f)$.

2) Geometric data of *obj1* and *obj2*: curvature forms $(K_1, K_2)$, metric tensors $(M_1, M_2)$, and connection forms $(T_1, T_2 = 0)$.

**Output:** An admissible path that links $p_0$ to $p_f$.

**Step 1:** Find a path $u_2(t) \in S_2$, $t \in [0, t_1]$, such that

$$u_2(0) = u_2^0 \text{ and } u_2(t_1) = u_2^f. \quad (18)$$

Let $u_1(t) \in S_1$ and $\psi(t)$, $t \in [0, t_1]$ be the induced trajectory of contact relative to *obj1* and the contact angle, respectively (i.e., the solution to (17)). At $t = t_1$, the contact point of *obj1* and the contact angle reach some intermediate values, denoted by

$$\hat{u}_1 = u_1(t_1) \text{ and } \hat{\psi} = \psi(t_1).$$

**Step 2:** Find a closed path $u_2(t) \in S_2$, $t \in [t_1, t_2]$, such that the induced contact trajectory of *obj1* has the property

$$u_1(t_1) = \hat{u}_1 \text{ and } u_1(t_2) = u_1^f.$$

Let $\psi(t)$, $t \in [t_1, t_2]$, be the induced trajectory of the contact angle. At $t = t_2$, the angle of contact reaches some intermediate value denoted by

$$\tilde{\psi} = \psi(t_2), \quad \text{where } \psi(t_1) = \hat{\psi}.$$

**Step 3:** Let $\delta\psi = \psi^f - \tilde{\psi}$ be the desired holonomy angle. Find a closed path $u_1(t) \in S_1$, $t \in [t_2, t_f]$, such that 1) the induced trajectory $u_2(t) \in S_2$, $t \in [t_2, t_f]$, is also *closed* and 2) the Gaussian curvature integral over the region bounded by $u_1$ is equal to the desired holonomy angle.

**Output:** Return the path $(u_1(t), u_2(t), \psi(t)) \in P$, $t \in [0, t_1] \cup [t_1, t_2] \cup [t_2, t_f]$, which is the union of the paths found in Steps 1, 2, and 3.

**Remark 4.2:** The desired contact point $u_2^f$ of *obj2* is achieved in Step 1. Then, using a closed curve relative to *obj2* in Step 2 the desired contact point $u_1^f$ of *obj1* is realized without sacrificing the desired contact point of *obj2*. Finally, in Step 3, using a closed curve relative to *obj1*, which also includes a closed curve relative to *obj2*, the desired contact angle is realized.

We now use the example of a unit ball on the plane to illustrate the algorithm. Clearly, Step 1 can be easily done using existing techniques in robot motion planning [4], [15]. Steps 2 and 3 are carried out as follows:

**Step 2A**: Let $\hat{u}_1$ and $u_1^f$ be the two contact points of *obj1*. We wish to construct a *closed path* $u_2(t)$, $t \in [t_1, t_2]$, in the plane so that the induced contact trajectory $u_1(t)$, $t \in [t_1, t_2]$, of $S^2$ links $\hat{u}_1$ to $u_1^f$.

**Lemma 4.1**: *Let $\hat{u}_1$ and $u_1^f$ be exactly $\pi/2$ distance apart in the unit sphere $S^2$. Then, the square of side length $\pi/2$, shown in Fig. 6 will induce a contact trajectory $u_1$ which links $\hat{u}_1$ to $u_1^f$.*

**Proof:** We need to demonstrate that the square has the desired features. Label the point $\hat{u}_1$ and $u_1^f$ in the sphere by $A'$ and $B'$, respectively, as shown in Fig. 6. $d(A', B') = \pi/2$. There exists a unique geodesic, i.e., an arc of the great circle, that connects $A'$ to $B'$. The great circle will be called the equator. Let $A$ denote the initial point of contact in the plane. Thus tracing the geodesic from $A'$ to $B'$ induces a straight line in the plane with endpoint $B$, and $d(B, A) = \pi/2$ (by arc length constraint). Going from the point $B$ to the point $C$ in the plane is equivalent to going from the point $B'$ to the north pole $C'$ in the sphere. Note that $\measuredangle(ABC)$ and $\measuredangle(A'B'C')$ are both right angles. Now, tracing the straight line from $C$ to $D$ in the plane induces a curve in the sphere which ends at the starting point $A'$. Consequently, by closing the curve in the plane with a straight line joining $D$ to $A$, we have arrived at the point $B'$ in the sphere. This shows that the square indeed induces a curve in the sphere which has a net incremental distance $\pi/2$. This is called a Lie bracket motion. □

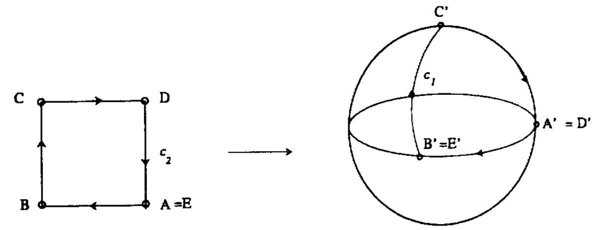



Fig. 6. A Lie bracket motion.

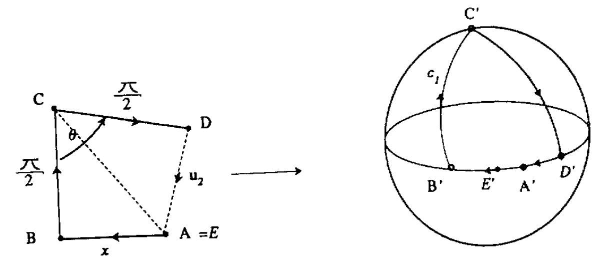



Fig. 7. A (general) Lie bracket motion.

We now return to the more general case.

**Step 2B**: By Lemma 4.1, we may assume that $d(\hat{u}_1, u_1^f) < \pi/2$. Otherwise, Lemma 4.1 can be applied repeatedly until some intermediate point which is less than $\pi/2$ distance away from $u_1^f$ is reached. Let $l = d(\hat{u}_1, u_1^f) < \pi/2$ be the distance of these two points. We wish to construct a closed curve $u_2(t)$, $t \in [t_1, t_2]$, in the plane such that the induced contact trajectory $u_1(t)$, $t \in [t_1, t_2]$, has an incremental distance $l$ along the geodesic connecting $\hat{u}_1$ to $u_1^f$. We propose to use for $u_2$ the closed curve $ABCDE$ shown in Fig. 7, where $x = d(A, B)$ is to be determined, $d(B, C) = d(C, D) = \pi/2$, and

$$\theta = 2 \tan^{-1} \frac{x}{\pi/2}.$$

We would like to show that for some choice of $x$, the closed curve $ABCDE$ will induce a curve $u_1(t)$, $t \in [t_1, t_2]$, in the sphere that links $\hat{u}_1$ to $u_1^f$. First, by tracing the straight line from $A$ to $B$ and then to $C$ induces a curve in the sphere which starts at $A'$, passes through $B'$, and then comes to the north pole $C'$. Note that $d(B', A') = x$ and $\measuredangle(A'B'C') = 90°$. Going down from $C$ to $D$ with an angle $\theta$ and by a distance $\pi/2$ is equivalent to going down in the sphere from $C'$ to some point $D'$ at the equator. Clearly, $d(B', D') = \theta$. Now, connect $D$ to $A$ by a straight line, and we claim that 1) $\measuredangle CDA = 90°$ and 2) $d(A, D) = x$. To see this, note that by definition $\measuredangle ACD = \theta/2$ and $AC$ is common to both the triangles $\Delta ABC$ and $\Delta ACD$. Thus they must be congruent triangles and the claim follows. Hence, by tracing the straight line from $D$ back to $A$ in the plane, we have followed the equator from $D'$ to some point $E'$, and $d(E', D') = x$. With

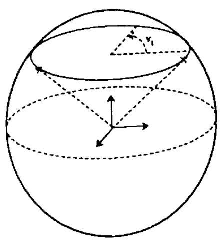


Fig. 8. Another Lie bracket motion.

$u_2$ being the closed curve $ABCDE$ for some choice of $x$, the induced curve $u_1$ in the sphere has its starting point $A'$ and its ending point $E'$, where $d(E', A')$, the net incremental distance, is a function of $x$. Let $f(x) = d(E', A')$. It is not hard to see that

$$f(x) = 2x - \theta = 2x - 2 \tan^{-1} \frac{x}{\pi/2}.$$

The hope is to find an $x$, if possible, that solves the equation

$$f(x) \stackrel{?}{=} l. \tag{19}$$

*We claim that there exists a unique x that solves* (19). To show this, note that $f(0) = 0$ and $f(\pi/2) = \pi/2 > l$. Thus solutions exist. For the uniqueness part, we compute the derivative of $f(x)$, which is given by

$$f'(x) = 2 - 2\frac{2/\pi}{1 + \frac{4x^2}{\pi^2}} = \frac{2 - 2/\pi + 4x^2/\pi^2}{1 + 4x^2/\pi^2} > 0.$$

Thus $f(x)$ is a monotone function and the solution to (19), denoted by $x^*$, is unique! Consequently, the curve $ABCDE$, with $d(B, A) = x^*$, has all the desired features.

**Step 3′**: We wish to find a *closed path* $u_1(t)$, $t \in [t_2, t_f]$, in $S^2$ such that 1) *the induced path* $u_2(t)$, $t \in [t_2, t_f]$, *in the plane is also closed* and 2) $u_1$ *has a desired holonomy angle* $\delta\psi$. We may assume that $0 < -\delta\psi < 2\pi$. Consider the latitude circle with $u_1(t) = u_1(0)$, and $v_1(t) = v_1(0) + t$, $t \in [t_2, t_2 + 2\pi]$, see Fig. 8. We claim that 1) *the induced trajectory* $u_2$ *is also a circle* and 2) *the holonomy angle of* $u_1$ *ranges from* 0 *to* $2\pi$ *for* $0 < u_1(0) < \pi/2$. To see this, substitute the expression of

$$\begin{bmatrix} u_1(t) \\ v_1(t) \end{bmatrix}$$

into (16) and after some algebra, we get

$$\psi(t) - \psi(0) = -\sin u_1(0)t \stackrel{\triangle}{=} \alpha t, \qquad \alpha = -\sin u_1(0)$$

and

$$u_2(t) = \beta \cos(\alpha t + \psi_0) + \gamma_0$$
$$v_2(t) = -\beta \sin(\alpha t + \psi_0) + \delta_0$$
$$\gamma_0 = u_2(0) - \cos \psi_0 \cos u_1(0)/\alpha$$
$$\delta_0 = v_2(0) + \sin \psi_0 \cos u_1(0)/\alpha.$$

Thus we have

$$(u_2(t) - \gamma_0)^2 + (v_2(t) - \delta_0)^2 = \beta^2.$$

This shows the claim.

## V. Conclusion

The paper studied a fundamental problem in dexterous manipulation by a robot hand: *motion of two rigid bodies with rolling constraint*. A systematic procedure for deriving the configuration space of contact and the differential equation for the constraint has been presented. This approach is applicable to objects of arbitrary shapes and under any contact constraints. For example, one may use this formulation to study motion of two rigid bodies under sliding or a combination of sliding and rolling constraints.

An algorithm that determines the existence of an admissible path between two contact configurations has been given. First, the distribution generated by the two constrained vector fields is computed. One then checks to see if the distribution is nonsingular. If so, an admissible path exists between any two contact configurations.

It has also been shown that the path finding problem is equivalent to a nonlinear control problem. Thus existing results in nonlinear control theory can be used. A geometric algorithm that finds a path when one object is flat is given.

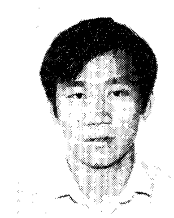
## Acknowledgment

The authors would like to thank Prof. S. Sastry and J. H. Lu for several valuable discussions, Prof. R. Brockett for pointing out two references in nonlinear control literature, J. Hauser for the help in using Macsyma to generate the ∇ for Example 3.2, and the reviewers for their excellent comments and suggestions.

## References

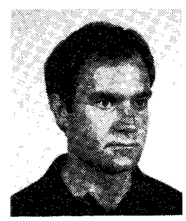
[1] M. Berry, "Classical adiabatic angles and quantal adiabatic phase," *J. Phys. A: Math. Gen.*, vol. 18, pp. 15–27, 1985.
[2] R. Brockett, "Control theory and singular Riemannian geometry," in *New Directions in Applied Mathematics*. New York, NY: Springer-Verlag, 1981, pp. 11–27.
[3] —, "On the rectification of vibratory motion," in *Proc. Microactuators and Micromechanism*, Salt Lake City, UT, 1988.
[4] J. F. Canny, *The Complexity of Robot Motion Planning*. Cambridge, MA: MIT Press, 1988.
[5] W. L. Chow, "Üeber Systeme von Linearen Partiellen Differentialgleichugen Erster Ordnung," *Math. Ann.*, vol. 117, no. 1, pp. 98–105, 1940.
[6] A. Cole, J. Hauser, and S. Sastry, "Kinematics and control of a multifingered robot hand with rolling contact," *IEEE Trans. Automat. Contr.*, vol. 34, no. 4, 1989.
[7] C. Cai and B. Roth, "On the spatial motion of rigid bodies with point contact," in *Proc. IEEE Int. Conf. Robotics and Automation*, pp. 686-695, 1987.
[8] C. Frohlich, "Do springboard divers violate angular momentum conservation," *Amer. J. Phys.*, vol. 47 no. 7, pp. 583–592, 1979.
[9] H. Goldstein, *Classic Mechanics*, 2nd ed. Reading, MA: Addison-Wesley, 1980.
[10] D. Greenwood, *Classical Dynamics*. Englewood Cliffs, NJ: Prentice-Hall, 1977.
[11] R. Hermann and A. J. Krener, "Nonlinear controllability and observability," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 728–740, 1977.
[12] R. Montgomery, J. Marsden, and T. Ratiu, "Reduction, symmetry, and Berry's phase in mechanics," Tech. Rep. PAM-438, Center for Pure and Applied Mathematics, Univ. of California, Berkeley, 1989.

[13] J. Kerr, "An analysis of multifingered hand," Ph.D. dissertation, Dep. Mech. Eng., Stanford Univ., Stanford, CA, 1985.
[14] W. Klingenberg, *A Course in Differential Geometry*. New York, NY: Springer-Verlag, 1978.
[15] D. E. Koditschek, "Exact robot navigation by means of potential functions: Some topological considerations," in *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 1-6, 1987.
[16] T. R. Kane and M. P. Scher, "A dynamical explanation of the falling cat phenomenon," *Int. J. Solid Structures*, vol. 5, pp. 663-670, 1969.
[17] Z. X. Li, P. Hsu, and S. Sastry, "On grasping and coordinated manipulation by a multifinger robot hand," *Int. J. Robotics Res.*, vol. 8, no. 4, 1989.
[18] Z. X. Li, "Motion planning with nonholonomic constraints," Master's thesis, Math. Dep., Univ. of California at Berkeley, Berkeley, CA 94720 (to appear in *IEEE Trans. Robotics Automat.*), 1990.
[19] P. F. Muir and C. P. Neuman, "Kinematic modeling of wheeled mobile robots," Tech. Rep., Robotics Inst., Carnegie-Mellon Univ., 1986.
[20] D. Montana, "Tactile sensing and kinematics of contact," Ph.D. dissertation, Div. Appl. Sci., Harvard Univ., 1986.
[21] R. S. Millman and G. D. Parker, *Elements of Differential Geometry*. Englewood Cliffs, NJ: Prentice-Hall, 1978.
[22] J. Marsden and T. Ratiu, *Mechanics and Symmetry*. Berkeley, CA: Univ. Calif., 1989.
[23] Panasonic Technical Reference, "Ultrasonic motor," Tech. Rep., Electric Motor Div., Matsushita Electric Industrial Co., Ltd, Japan, 1988.
[24] E. D. Sontag, "Integrability of certain distributions associated to actions on manifolds and an introduction to Lie-algebra control," Tech. Rep., Dep. Math., Rutgers Univ., Rep. SYCON-88-04, 1988.
[25] —, "Some complexity questions regarding controllability," Tech. Rep., Dept. Math., Rutgers Univ., Rep. SYCON-88-06, 1988.
[26] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vols. I and II. Boston, MA: Publish or Perish, 1974.
[27] J. A. Thorpe, *Elementary Topics in Differential Geometry*. New York, NY: Springer-Verlag, 1978.
[28] C. Von Westenholz, *Differential Forms in Mathematical Physics*. Amsterdam, The Netherlands: North-Holland, 1981.



**Zexiang Li** was born in a village in the Hunan province, People's Republic of China, in September of 1961. He attended the Central-South University of Technology during 1978, received the B.S. degrees in economics and in electrical engineering (with honors) from Carnegie-Mellon University in 1983, the M.S. degree in electrical engineering and computer sciences in 1985, the M.A. degree in mathematics, and the Ph.D. degree in electrical engineering and computer sciences in 1989, all from the University of California at Berkeley.

He is currently a research scientist in the Artificial Intelligence Laboratory, MIT, Cambridge, MA, working on legged locomotion vehicles. He will be an assistant professor at the Courant Institute, New York University, starting in the Spring of 1990. His research interests include analysis, planning and control of nonlinear systems, e.g., dextrous robot hands, legged locomotion vehicles, space robots, and robot manipulators, as well as machine vision.



**John Canny** was born in Adelaide, Australia, in 1958. He received the B.Sc. and B.E. (Hons) degrees from Adelaide University in 1980 and 1981, respectively. He then studied at the Massachusetts Institute of Technology, and received the S.M. and Ph.D. degrees in computer science in 1983 and 1987 respectively.

He has been an assistant professor in the Computer Science Division at the University of California, Berkeley, since August 1987. His research interests are motion planning for assembly and mobile robots, computation geometry, algebraic algorithms, and geometric modeling. He is an Associate Editor of *Robotics Reviews*. He received the ACM Doctoral Dissertation Award in 1987, a David and Lucile Packard Foundation Fellowship in 1988, and a Presidential Young Investigator Award in 1989.