Output feedback control with input saturations: LMI design approaches

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Abstract

This paper addresses the control of linear systems with input saturations. We seek a controller that guarantees for the closed loop system: (i) stability for a given polytope of initial conditions, (ii) a prescribed weak $\mathcal{L}_2$ gain attenuation between inputs and outputs of interest. Two approaches are proposed based on: (i) ensuring that the controller never saturates: the obtained controller is Linear Time Invariant (LTI), (ii) ensuring absolute stability against the saturations: the controller is then Linear Time Varying (LTV). Existence conditions for these two control structures can be cast as (convex) optimization problems over Linear Matrix Inequalities. At last, using numerical experiments, we compare both approaches. In this numerical examples, the LTI controller presents some advantages.

Keywords Control saturation, output feedback control, absolute stability, Linear Matrix Inequalities.

1 Introduction

Considered problem. Saturations are always present at the input of real plants. Actuator saturations can dramatically degrade the performance or even destabilize the closed loop system. A classical approach to deal with them is the anti wind-up design [13, 17]. A controller is first designed ignoring the input saturations. Then, it is modified to take into account the effects of the non linearities [5]. The last step is the verification of the stability and the performance of the closed loop system. It is thus a "trial and error" approach.

Avoiding the "trial and error" approach is a challenging problem. In the case of saturating actuators, the design of output feedback control laws ensuring a certain level of performance by a low complexity algorithm is still an open problem. In this paper, we propose a low complexity algorithms which allow to design output feedback controllers ensuring a certain level of performance in this case. The proposed conditions reduce to convex optimization problem involving Linear Matrix Inequality constraints whose complexity is mild.

Related works. Avoiding the "trial and error" approach is a philosophy which was adopted in [2] to obtain a design method for positive real plants with arbitrary input nonlinearities. More general results were proposed in [15]: a linear controller design method is proposed to minimize a quadratic performance criterion for the control of a plant with input nonlinearities. Their method also addresses the absolute stabilization problem. Unfortunately, the obtained conditions are not convex. Non convex process design was also proposed in [25], where the methodology is based on the Popov Criteria.

In [28], the process design is proposed in order to obtain control laws ensuring (semi-global) stability and the rejection of input disturbance. It requires the solution of a parameterized Algebraic Riccati Equation, an infinite dimensional problem which is numerically difficult to handle. A related approach is in [22]. In this paper, the stabilisation and perturbation rejection are considered: nevertheless, no performance level is guaranteed. Focusing in state feedback control, the basic idea is to tune a control law $u = K(r)x$ using the parameter r: r closed to 0 correspond to a low-gain control, ensuring a large domain of attraction; r closed to 1 correspond to a high-gain control, ensuring a small domain of attraction. In order to design $K(r)$, several parameters are to be fixed without precise choice rules allowing to improve closed loop performance. Furthermore, the one-line adaptation of r is quite complex. In fact, both approaches suffers from the complexity of the process design and of the obtained control law.

In [18], control laws are designed in order to ensure disturbance attenuation of bounded energy disturbances on the control input. The process design is quite direct when local stability is desired: in the case of semi-global stability the approach is based on the resolution of a parameterized Algebraic Riccati Equation. Unfortunately, only the state feedback case is considered.

Another very interesting approach was proposed by [35]. In [35], a possible rigorous definition of the anti wind-up problem is proposed. The purpose is to modify a previously designed control law (ignoring saturation) in order to ensure that an input saturation does not lead to a strong difference between the plant with and without input saturation. The requirement is quite weak (L_2 stability). Furthermore, the anti wind-up control is not optimized in order to reduce performance degradation. In fact, the state space model of the controlled plant is introduced in the anti wind-up modification. Under the assumption that the model plant is perfect, the closed-loop system can be equivalently rewritten as the feedforward connection of two subsystems. The anti wind-up modification is then developed in order to stabilize both sub-systems. Note that if this strong assumption is not satisfied, a feedback loop is introduced, possibly leading to instability.

Proposed approaches. In this paper, we consider two approaches which directly give an output feedback controller, avoiding the "trial and error" approach and a priori ensuring a certain level of performance. The first approach consists in avoiding actuator saturation. It is just a modification of previously published results (see *e.g.*[8, 6]). From a certain point of view, this approach extends the proposed one in [14]. In this paper, state feedback control design is design by a "trial and error" process in order to ensure stability. The approach in [3] is related: a nonlinear optimization based process design is proposed to tune a stabilizing state feedback control law. Our approach is in fact the extension of the method design proposed in [4] from the state feedback case to the output feedback case. The second one is based on absolute stability. In this approach, the problem is recast as a gain scheduling controller design problem and the LMI framework is applied [24, 1, 32]. It is the first contribution of this paper. The second contribution is a comparison between these two approaches. These approaches were previously presented the 1997 conference paper [30]. Note that related methods was further developed in, *e.g.* the conference papers [37, 23]. In [33, 34], a similar idea is used in order to design a stabilizing state feedback control law. The representation of the gain scheduled control system is nevertheless different: instead of a Linear Fractional Transformation representation (as we consider), they consider a polytopic representation.

We propose convex formulations for both approaches. In fact, this nice result is possible due to sufficient conditions. As a consequence, these approaches could present a certain conservatism with respect to the considered problem. Nevertheless, our main purpose is to achieve a trade-off between this possible conservatism and the complexity of our approach. Our most important constraint is to obtain methods which allow to design low-complexity control laws (in order to reduce the real time computation burden), using low complexity methodologies, that is, readily solvable using a computer. As a consequence, we propose *design process* which boils down to *finite dimensional, convex optimization* problems. Non convex formulations or infinite dimensional formulations are not considered. Furthermore, we only consider the *output feedback control law* design in order to ensure a *performance* specification in addition to local stability. The complexity of the proposed control law is mild, allowing an easy on-line implementation.

We are interested in finding controllers ensuring stability for a specified closed loop system state space domain. The assumption of non global stability is rather mild: in fact, models of physical systems are only representative for a bounded part of the state space. Furthermore, an unstable plant with input saturations is not globally null controllable [27]. For instance, in the paper [19], it is explained that only semi global stability can be achieved by linear control for some classes of unstable plants.

In [20], a class of linear systems (neutrally stable systems) is proved to be globally stabilizable by bounded state and output feedback control laws of a certain structure. No process design is proposed. Note that, from [29], our output feedback LTV controller can be rewritten in a closely related form, when only stability is considered. Such a fact emphasizes the interest of LTV control law structure.

Underlying ideas. Our approach is as follows. The main fact is that we are able to guarantee that the obtained controller has a control magnitude bounded by a given u_{max}. If u_{max} is less than the saturation level u_{sat} then the actuator never saturates and the controller is LTI. This controller is referred to as a *non saturating controller.* If $u_{max} > u_{sat}$, only a portion of the saturation is "excited" (see figure 1). This information can be used to guarantee absolute stability against the "excited" portion of the nonlinearity. This information is used to reduce the conservatism of the usual absolute stability approach. In this case, the controller has an LTV structure. In the sequel, this controller is referred to as

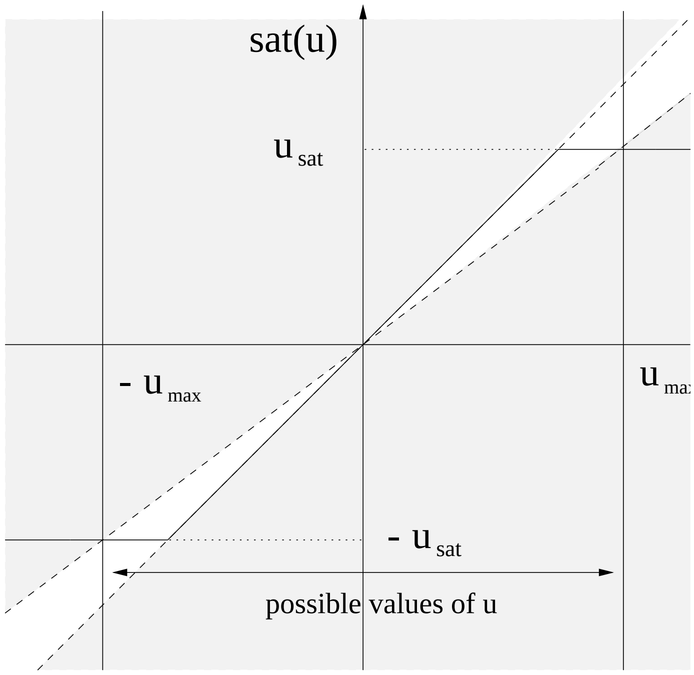

Figure 1: "Excited" portion of the saturation

a *saturating controller*. In fact, as in the gain scheduling approach [24, 1, 32], the control law gains are tuned by a time-varying parameter, which is readily computed from the control input u.

In this paper, we compare potentials of both approaches. To the purpose, we consider a system submitted to perturbations of finite $\mathcal{L}_2$ norm (that is, energy) and with bounded initial conditions. For this class of problems, we are able to design a controller which excites only a specified portion of the nonlinearity. Our approach is less conservative than directly applying absolute stability. This idea is quite usual in practical engineering. The new fact is that we are able to *guarantee* that the controller excites a given portion of the nonlinearity. The extent of this portion, measured by u_{max}, can be considered as a design parameter.

Paper outline. The paper is organized as follows. In section 2, the considered problem is briefly presented. The next section is a discussion of the two proposed controller structures. Existence conditions are provided in the section 4. The *saturating controller* proposed in this paper is then compared with a previous approach. Explicit expressions for both controllers are given in the section 6. Numerical experiments allow to compare the two designs in the last section. The proofs are derived by specializing existing general results [8, 32] to our control problem. The main purpose is to discuss their application to control with saturations.

Notations. $\mathcal{L}_2$ denotes the set of square integrable functions from $\mathbf{R}$ to $\mathbf{R}^p$.
I_r and 0_r denote the identity and the zero matrices of $\mathbf{R}^{r\times r}$, with I_0 (or 0_0) empty. The subscript is omitted when it is evident from the context. Let U be a full rank $r\times n$ real matrix with $r < n$. $U^\perp$ denotes an orthogonal complement of U, i.e., $UU^\perp = 0$ and $\begin{bmatrix} U^T & U^\perp \end{bmatrix}$ is of maximal rank. **Co** denotes the convex hull. For a given integer k, $\mathcal{S}(k)$ denotes the following set:

$$\mathcal{S}(k) = \left\{ S \in \mathbf{R}^{k\times k} \mid S = \begin{bmatrix} s_1 & 0 & \dots & \dots & 0 \\ 0 & s_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & s_k \end{bmatrix} \right\}$$

If M is a gain matrix split as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and Δ an operator then the operator $\mathcal{F}_l(M,\Delta)$ is defined by: $\mathcal{F}_l(M,\Delta) \triangleq M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}$.

2 Problem statement

We consider a continuous-time system, with input saturations:

$$\begin{array}{rclcccl} \dot{x} & = & Ax & + & B_w w & + & \sum_{i=1}^{n_u} B_{u_i}\mathbf{sat}(u_i) \\ z & = & C_z x & + & D_{zw} w & + & \sum_{i=1}^{n_u} D_{zu_i}\mathbf{sat}(u_i) \\ y & = & C_y x & + & D_{yw} w & & \end{array} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^{n_u}$ the command input, $y \in \mathbf{R}^{n_y}$ the measured output, $z \in \mathbf{R}^{n_z}$ the output of interest and $w \in \mathbf{R}^{n_w}$ the disturbance input. The saturations are defined as follows:

$$\begin{array}{rclrcl} \mathbf{sat}(u_i) & = & -u_{i\,sat} & u_i & \leq & -u_{i\,sat} \\ \mathbf{sat}(u_i) & = & u_i & |u_i| & \leq & u_{i\,sat} \\ \mathbf{sat}(u_i) & = & u_{i\,sat} & u_i & \geq & u_{i\,sat}. \end{array}$$

In the sequel, without any loss of generality, we assume that $u_{1\,sat} = \cdots = u_{n_u\,sat} = u_{sat}$. Let us define $B_u = \begin{bmatrix} B_{u_1} \cdots B_{u_{n_u}} \end{bmatrix}$ and $D_u = \begin{bmatrix} D_{u_1} \cdots D_{u_{n_u}} \end{bmatrix}$.

Consider a polytope of initial conditions $\mathcal{P}$ containing zero, described by its vertices: $\mathcal{P} = \mathbf{Co}\{v_1, \ldots, v_p\}$. Furthermore, consider that the disturbance input has an $\mathcal{L}_2$ norm is less than one, that is, for every $T > 0$,

$$\int_0^T w(t)^T w(t)dt \leq 1.$$

It can be interpreted as a bound over the energy of the input w. Given this set of initial conditions and input disturbances, we want to devise a dynamic controller which ensures for the closed loop system the following specifications: (i) stability, (ii) "weak" (γ, β) $\mathcal{L}_2$-gain between the input disturbance w and the output of interest z, that is, for a given $\gamma > 0$ and $\beta > 0$ and for all positive T:

$$\int_0^T z(t)^T z(t)dt \leq \gamma^2 \int_0^T w(t)^T w(t)dt + \beta(x_0)$$

with $\beta(x_0) \leq \beta$ and $\beta(0) = 0$.

The last requirement is now interpreted.

- If $x_0 = 0$ then $\int_0^T z(t)^T z(t)dt \leq \gamma^2 \int_0^T w(t)^T w(t)dt$. The closed loop system has then an $\mathcal{L}_2$ gain between w and z less than γ.
- If $w = 0$ then $\int_0^T z(t)^T z(t)dt \leq \beta$. For an initial condition taken in $\mathcal{P}$, the $\mathcal{L}_2$-norm of z is bounded by $\sqrt{\beta}$. It is related to the definition of an H_2 norm constraint for the set $\mathcal{P}$ of initial conditions.

In the sequel, such a closed loop system will be referred as $\{\mathcal{P}, \gamma, \beta\}$.

3 Controller structures

The input nonlinearity $\mathbf{sat}(u_i)$ can be considered as a time varying uncertainty on the input u_i, that is:

$$\mathbf{sat}(u_i) = (1 - \delta_i(t))u_i(t) \text{ with } 0 \leq \delta_i(t) < 1$$

where $\delta_i(t)$ is an increasing function of $u_i(t)$ with $\delta_i(t) \to 1$ when $u_i(t) \to +\infty$. This interpretation of a memoryless nonlinearity was already noted by [26]. In this paper, it is stressed that ensuring stability against a memoryless nonlinearity is equivalent to ensuring stability against a time varying uncertainty.

For initial conditions in $\mathcal{P}$ and a disturbance input w with an $\mathcal{L}_2$ norm less than one, the conditions proposed in the section 4 ensure that the modulus of $u_i(t)$ is bounded by a given

u_{max}. If $u_{max} > u_{sat}$ then we know that $0 \leq \delta_i(t) \leq 1 - u_{sat}/u_{max} < 1$. As u_i is perfectly known and as u_{sat} is known too, it is possible to readily compute the uncertainty $\delta_i(t)$. If $u_{max} \leq u_{sat}$ then the input of the system never saturates ($\delta_i(t) = 0$). u_{max} is then a design parameter.

Let $\delta(t)$ be $[\delta_1(t), \cdots, \delta_{n_u}(t)]$. Two different approaches are proposed in the sequel. First, if, with $u_{max} = u_{sat}$, the specifications are respected then the controller does not saturate the input. The dynamic controller has the following form:

$$\begin{array}{rcll} \dot{\bar{x}} & = & \bar{A}\,\bar{x} & + \quad \bar{B}_y\, y, \quad \bar{x}(0) = 0, \\ u & = & \bar{C}_u\,\bar{x}. & \end{array} \tag{2}$$

This approach was already considered in [4, 6, 8] in more general contexts. Its application to our case is provided for the sake of completeness. Existence conditions of such a controller can be cast as an optimization problem. If the conditions hold, the matrices $\bar{A}$, $\bar{B}_y$ and $\bar{C}_u$ are computed using linear algebra. In the sequel, this controller is referred as a *non saturating controller*.

Second, if the specifications are respected with $u_{max} > u_{sat}$ then we seek the following controller:

$$\begin{array}{rcll} \dot{\bar{x}} & = & \bar{A}(\delta(t))\,\bar{x} & + \quad \bar{B}_y(\delta(t))\, y, \quad \bar{x}(0) = 0, \\ u & = & \bar{C}_u\,\bar{x}, & \end{array} \tag{3}$$

where $\bar{A}$ and $\bar{B}_y$ are (multi variable) rational functions of $\delta(t)$ and $\bar{C}_u$ a matrix. The gain of this controller is scheduled by the uncertainty $\delta(t)$. (For more information about the links between the saturated and the gain scheduled control, see the paper [29, 32].) As in the previous case, existence conditions can be cast as an optimization problem. If the conditions hold, the matrices $\bar{A}$, $\bar{B}_y$ and $\bar{C}_u$ are computed using linear algebra. In the sequel, this controller is referred as a *saturating controller*. Nevertheless, a less complex controller structure can be interesting, for instance, a linear time invariant controller similar to (2). This formulation leads to a non convex optimization problem more constrained than in the previous case. In the sequel, this controller is referred to as a *robust controller*.

4 Sufficient conditions for saturated control

The first theorem gives sufficient conditions ensuring the existence of a *non saturating controller* such that the closed loop system is $\{\mathcal{P}, \gamma, \beta\}$.

Theorem 4.1 *If there exist matrices $P, Q \in \mathbf{R}^{n\times n}$, $Y = \begin{bmatrix} Y_1^T & \cdots & Y_{n_u}^T \end{bmatrix}^T \in \mathbf{R}^{n_u\times n}$ and $Z \in \mathbf{R}^{n\times n_y}$ such that*

$$\left[\begin{array}{c|c} \begin{array}{c} A^TP+PA+C_z^TC_z \\ +ZC_y+C_y^TZ^T \end{array} & \begin{array}{c} PB_w+C_z^TD_{zw} \\ +ZD_{yw} \end{array} \\ \hline \begin{array}{c} {B_w}^TP+{D_{zw}}^TC_z \\ +D_{yw}^TZ^T \end{array} & D_{zw}^TD_{zw}-\gamma^2 I \end{array}\right] < 0 \tag{4}$$

$$\left[\begin{array}{c|c} \begin{array}{c} AQ+QA^T+B_uY \\ +Y^TB_u^T+B_wB_w^T \end{array} & \begin{array}{c} QC_z^T+Y^TD_{zu}^T \\ +B_wD_{zw}^T \end{array} \\ \hline \begin{array}{c} C_zQ+D_{zu}Y \\ +D_{zw}B_w^T \end{array} & D_{zw}D_{zw}^T-\gamma^2 I \end{array}\right] < 0 \tag{5}$$

$$v_j^TPv_j \leq \beta, \quad \textit{for } j = 1, \ldots, p, \tag{6}$$

$$u_{sat} \begin{bmatrix} \frac{\gamma^2}{\gamma^2+\beta}I & 0 & 0 \\ 0 & Q & \gamma I \\ 0 & \gamma I & P \end{bmatrix} + \begin{bmatrix} 0 & Y_i & 0 \\ Y_i^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0, \quad \textit{for } i = 1, \ldots, n_u. \tag{7}$$

then there exists a control law of the form (2) such that

- *the closed loop system is stable for every initial condition in the polytope $\mathcal{P}$ and disturbance input with an $\mathcal{L}_2$ norm less than one,*
- *the closed loop system is weak (γ, β) $\mathcal{L}_2$-gain between the input disturbance w and the output of interest z.*

Proof See Appendix, section A. □

One just need to find the matrices P, Q, Y and Z such that conditions (4), (5), (6) and (7) are satisfied. This problem is a convex optimization problem involving Linear Matrix Inequality constraints. It can be solved extremely efficiently, using the free ware code SP [36] and its Matlab interface `LMITOOL` [7].

We now give an interpretation of the conditions of the previous theorem. Condition (4) ensures what there exists a linear observer:

$$\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x} + B_uu - P^{-1}Z(y-\hat{y}) \\ \hat{y} & = & C_y\hat{x} \\ \hat{z} & = & C_z\hat{x} + D_{zu}u \end{array}$$

for the system:

$$\begin{array}{rclclcl} \dot{x} & = & Ax & + & B_w w & + & B_u u \\ y & = & C_y x & + & D_{yw} w & & \\ z & = & C_z x & + & D_{zw} w & + & D_{zu} u \end{array}$$

such that the $\mathcal{L}_2$ gain between w and $z - \hat{z}$ is less than γ. Condition (5) ensures that the linear state feedback controller $u = YQ^{-1}x$ such that the linear closed loop system:

$$\begin{array}{rclcl} \dot{x} & = & (A + B_u K)x & + & B_w w \\ z & = & (C_z + D_{zu} K)x & + & D_{zw} w \end{array}$$

is stable and has an $\mathcal{L}_2$ gain between w and z less than γ. Furthermore, condition (7) ensures that the control input u never saturates for the considered set of initial conditions and disturbances.

The second theorem gives sufficient conditions ensuring the existence of a *saturating controller* such that the closed loop system is $\{\mathcal{P}, \gamma, \beta\}$.

Theorem 4.2 *If there exist matrices* $P, Q \in \mathbf{R}^{n \times n}$, $S, T \in \mathcal{S}(n_u)$, $Y = \left[\begin{array}{ccc} Y_1^T & \cdots & Y_{n_u}^T \end{array}\right]^T \in \mathbf{R}^{n_u \times n}$ *and* Z_1, Z_2 *and a scalar* $u_{max} > u_{sat}$ *such that:*

$$u_{max} \left[\begin{array}{c|c|c} \begin{array}{c} A^T P + PA + C_z^T C_z \\ + Z_1 C_y + C_y^T Z_1^T \end{array} & \begin{array}{c} PB_w + C_z^T D_{zw} \\ + Z_1 D_{yw} \end{array} & \begin{array}{c} PB_u + C_z^T D_{zu} \\ + C_y^T Z_2^T \end{array} \\ \hline \begin{array}{c} B_w{}^T P + D_{zw}{}^T C_z \\ + D_{yw}^T Z_1^T \end{array} & D_{zw}^T D_{zw} - \gamma^2 I & D_{zw}{}^T D_{zu} + D_{yw}^T Z_2^T \\ \hline \begin{array}{c} B_u^T P + D_{zu}^T C_z \\ + Z_2 C_y \end{array} & D_{zu}^T D_{zw} + Z_2 D_{yw} & D_{zu}^T D_{zu} - 2S \end{array}\right] - u_{sat} \left[\begin{array}{ccc} 0 & 0 & C_y^T Z_2^T \\ 0 & 0 & 0 \\ Z_2 C_y & 0 & 0 \end{array}\right] < 0 \tag{8}$$

$$u_{max} \left[\begin{array}{c|c|c} \begin{array}{c} AQ + Y^T B_u^T \\ + QA^T + B_u Y \\ + B_w B_w^T \end{array} & \begin{array}{c} QC_z^T + B_w D_{zw}^T \\ + Y^T D_{zu}^T \end{array} & -Y^T + B_u T \\ \hline \begin{array}{c} C_z Q + D_{zu} Y \\ + D_{zw} B_w^T \end{array} & D_{zw} D_{zw}^T - \gamma^2 I & D_{zu} T \\ \hline -Y + TB_u^T & TD_{zu}^T & -2T \end{array}\right] + u_{sat} \left[\begin{array}{ccc} 0 & 0 & Y^T \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{array}\right] < 0 \tag{9}$$

$$v_j^T P v_j \leq \beta, \text{ for } j = 1, \ldots, p, \quad \left[\begin{array}{cc} S & \gamma I \\ \gamma I & T \end{array}\right] \geq 0 \tag{10}$$

$$u_{max} \left[\begin{array}{ccc} \frac{\gamma^2}{\gamma^2 + \beta} I & 0 & 0 \\ 0 & Q & \gamma I \\ 0 & \gamma I & P \end{array}\right] + \left[\begin{array}{ccc} 0 & Y_i & 0 \\ Y_i^T & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \geq 0, \text{ for } i = 1, \ldots, n_u \tag{11}$$

then there exists a control law of the form (3) such that:

- *the closed loop system is stable for every initial condition in the polytope $\mathcal{P}$ and disturbance inputs with an $\mathcal{L}_2$ norm less than one,*
- *the closed loop system is weak (γ, β) $\mathcal{L}_2$-gain between the input disturbance w and the output of interest z.*

Proof See Appendix, section B. □

For a fixed u_{max}, the inequalities (10), (8), (9) and (11) also linearly depend on P, Q, S, T, Y, Z_1 and Z_2. The function which gives an u_{max} such that there exist P, Q, S, T, Y, Z_1 and Z_2 which verify the conditions (10), (8), (9) and (11) are satisfied is then quasi convex. Existence conditions of a *saturating controller* can be cast as a Linear Matrix Inequality optimization problem associated with a line search over the design parameter u_{max}.

We just give an interpretation of the conditions of the previous theorem. Condition (8) ensures what there exists an observer:

$$\begin{array}{rcl} \dot{\hat{x}} &=& A\hat{x} + \sum_{i=1}^{n_u} B_{u_i}\mathbf{sat}(u_i) - \left(P^{-1}Z_1 + \sum_{i=1}^{n_u} B_{u_i}S^{-1}Z_2\left(1 - \frac{\mathbf{sat}(u_i)}{u_i}\right)\right)(y - \hat{y}) \\ \hat{y} &=& C_y\hat{x} \\ \hat{z} &=& C_z\hat{x} + \sum_{i=1}^{n_u} D_{zu_i}\mathbf{sat}(u_i) \end{array}$$

for the system (1) and such that the $\mathcal{L}_2$ gain between w and $z - \hat{z}$ is less than γ. Condition (9) ensures that the linear state feedback controller $u = YQ^{-1}x$ stabilise system (1) and has an $\mathcal{L}_2$ gain between w and z less than γ. Furthermore, condition (11) ensures that the control input u has a modulus less than u_{max} for the considered set of initial conditions and disturbances.

A discussion about the conditions obtained for both approaches is of interest. Consider the case when $D_{zu} = 0$. In this case, if condition (5) holds then there exists a scalar $\epsilon > 0$ small enough such that with $S = \frac{1}{\epsilon}I$, the condition (8) holds. The condition:

$$\left[\begin{array}{cc} S & \gamma I \\ \gamma I & T \end{array}\right] \geq 0$$

then holds if the smallest eigenvalue T is just constrained to be bounded from below, by $\epsilon\gamma^2$. Now, consider the condition (11) in the theorem 4.2. As $u_{max} > u_{sat}$, this condition is relaxed compared to the corresponding condition (7) in the theorem 4.1. But, if you consider the condition (9) in the theorem 4.2, it is clearly harder than the corresponding condition in the theorem 4.1. To sum up, compared to the conditions of the theorem 4.1, the theorem 4.2 contains one relaxed condition and one tightened condition. At first glance, it is unclear which method performs better. Numerical experiments are proposed in the section 7.

5 Relation between the second approach and previous results

The second approach pertains to the class of absolute stabilization methods. For instance, the method proposed in the paper [15] also relies on absolute stability idea. In fact, this last method can be adapted to the approach and the problem considered in this paper. In [15], the authors consider a global stabilization problem: the *robust controller* must ensure the stability against the whole nonlinearity. In the present paper, the obtained conditions are also non convex and guarantee the existence of an LTI controller ensuring stability for a specified state space domain. This problem is closely related to the robust control problem in the context of uncertain linear systems. The results are summarized in the following theorem:

Theorem 5.1 *If there exist matrices $P, Q \in \mathbf{R}^{n\times n}$, $S, T \in \mathcal{S}(n_u)$, $Y \in \mathbf{R}^{n_u \times n}$ and Z_1, Z_2 and a scalar $u_{max} > u_{sat}$ such that :*

$$\begin{aligned} &v_j^T P v_j \leq \beta, \quad for \;\; j = 1, \ldots, p, \\ &S = \gamma^2 T^{-1}, \quad S > 0 \quad and \quad (8), \quad (9), \quad (11). \end{aligned}$$

then there exist a robust controller *such that: (i) the closed loop system is stable for every initial condition in the polytope $\mathcal{P}$ and disturbance input with an $\mathcal{L}_2$ norm less than one. (ii) the closed loop system is weak (γ, β) $\mathcal{L}_2$-gain between the input disturbance w and the output of interest z.*

Proof See Appendix, section C. □

The conditions of the theorem 5.1 are the conditions of the theorem 4.2 except one: now, the inequality

$$\begin{bmatrix} S & \gamma I \\ \gamma I & T \end{bmatrix} \geq 0$$

is tightened to $S = \gamma^2 T^{-1}$, with $S > 0$. The conclusion is that the obtained performances with the *saturating controller* are at least the same or even better than with the controller considered in this section.

6 Explicit expressions of both controllers

Now, assume that the previous existence conditions (4), (5), (6), (7) (for the *non saturating controller*) hold. Then the controller matrices $\bar{A}$, $\bar{B}_y$ and $\bar{C}_u$ of the control law (2) can be obtained from the matrices P, Q, Y and Z by the following expressions:

$$\begin{aligned}
\bar{C}_u &= -YQ^{-1}M^{-T}, \\
\bar{B}_y &= M^{-1}Z, \\
\bar{A} &= M^{-1}\Bigg(PA + \gamma^2 A^T Q^{-1} + PB_u Y Q^{-1} + ZC_y \\
&\quad - \begin{bmatrix} (PB_w + ZD_{yw})^T \\ C_z \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I & D_{zw}^T \\ D_{zw} & -I \end{bmatrix}^{-1} \begin{bmatrix} \gamma^2 B_w^T Q^{-1} \\ C_z + D_{zu} Y Q^{-1} \end{bmatrix}\Bigg) M^{-T}.
\end{aligned}$$

where M is obtained from $P - \gamma^2 Q^{-1} = MM^T$. For a proof, see Appendix, section A.

If the conditions for the *saturating control* (3) hold then the parameter dependent state space matrices are given by:

$$\begin{aligned}
\bar{C}_u &= -YQ^{-1}M_1^{-T}, \\
\bar{B}_y(\delta) &= \bar{B}_y + \bar{B}_{\bar{p}}\Delta(t)\left(I - \bar{D}_{\bar{q}\bar{p}}\Delta(t)\right)^{-1}\bar{C}_{\bar{q}}, \\
\bar{A}(\delta) &= \bar{A} + \bar{B}_{\bar{p}}\Delta(t)\left(I - \bar{D}_{\bar{q}\bar{p}}\Delta(t)\right)^{-1}\bar{D}_{\bar{q}y}
\end{aligned}$$

with $\bar{B}_y = M_1^{-1} Z_1$, $\bar{D}_{\bar{q}y} = M_2^{-1} Z_2$ and

$$\begin{bmatrix} \bar{A} & \bar{B}_{\bar{p}} \\ \bar{C}_{\bar{q}} & \bar{D}_{\bar{q}\bar{p}} \end{bmatrix} = \begin{bmatrix} M_1^{-1} & 0 \\ 0 & \frac{u_{max}}{u_{max}-u_{sat}} M_2^{-1} \end{bmatrix} \left(\left[\begin{array}{l} PA + \gamma^2 A^T Q^{-1} \\ +PB_u Y Q^{-1} + Z_1 C_y \\ \gamma^2 B_u^T Q^{-1} + (1 - u_{sat}/u_{max}) Z_2 C_y \\ -(1 - u_{sat}/u_{max}) S Y Q^{-1} \end{array} \right. \cdots \right.$$

$$\cdots \left. \begin{array}{c} PB_u \\ 2\gamma^2 T^{-1} \end{array} \right] - \begin{bmatrix} PB_w + Z_1 D_{yw}^T & C_z^T \\ Z_2 D_{yw} & D_{zu}^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & D_{zw}^T \\ D_{zw} & -I \end{bmatrix}^{-1} \cdots \tag{12}$$

$$\cdots \left. \begin{bmatrix} \gamma^2 B_w^T Q^{-1} & 0 \\ C_z + D_{zu} Y Q^{-1} & D_{zu} \end{bmatrix} \right) \begin{bmatrix} M_1^{-T} & 0 \\ 0 & M_2^{-T} \end{bmatrix}.$$

where

$$\Delta(t) = \begin{bmatrix} \delta_1(t) & 0 & \cdots \\ 0 & \ddots & \cdots \\ \cdots & \cdots & \delta_{n_u}(t) \end{bmatrix},$$

and M_1 (M_2) are such that $P - \gamma^2 Q^{-1} = M_1 M_1^T$ ($S - \gamma^2 T^{-1} = M_2 M_2^T$). For a proof, see Appendix, section B.

7 Numerical experiments

To compare these two methods, numerical experiments are performed. The LMI optimization problem was solved using the free ware code SP [36] with its Matlab interface `LMITOOL` [7] or the commercial Matlab toolbox `LMI Control Toolbox` [12].

Some results are exposed in this section. For a given system G, described by its state space representation matrices:

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array} \right],$$

a fixed performance level, that is, an initial condition polytope $\mathcal{P} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$, a fixed $\beta = 1$ and $\gamma = 1$, we try to compute the controller which achieves this performance level

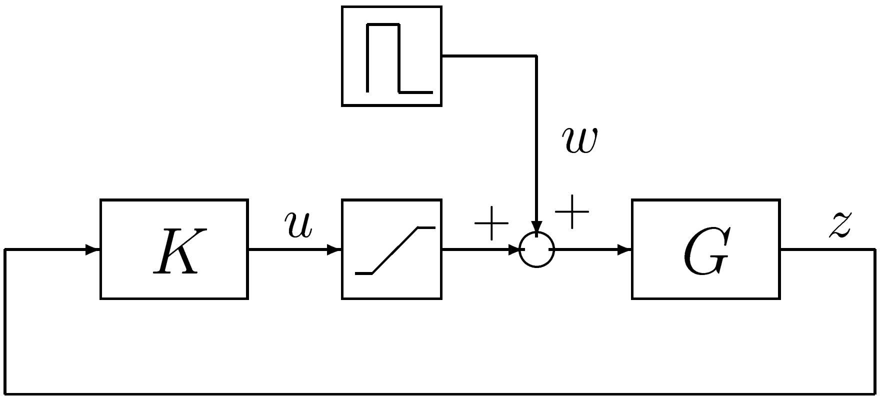

Figure 2: Closed-loop system with saturation

with the smallest possible u_{sat}. By applying the first approach, the obtained controller K, after simplifications, is given by

$$K(s) = -4.93\frac{s + 0.60}{s + 2.92}.$$

Note that $K(s)$ boils down to a lead controller. In this case, the application of the second method does not allow to improve the performance level γ obtained with *non saturating controllers*. So the *saturating controllers* are not provided. It is the main experimental conclusion: even if the *saturating controller* structure is more complex than the LTI structure and the existence conditions slightly different, the performance is not improved. Other numerical experiments confirm this conclusion. Pay attention to this conclusion. It is not claimed that performance is not improved when allowing saturation. It is claimed that the saturating controller obtained by our approach does not seem to improve the performance of the non saturating controller obtained by our approach. As discussed in section 4, this conclusion can not a priori be straightforwardly deduced from both problem formulation.

For the initial condition

$$x(0) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T$$

and for the input signal w (in dotted dashed), figure 3 shows the command input u (in dashed) and the system output z (in plain). The command input u never exceeds 2.01 and the energy of the output z 0.413 is less than $\sqrt{\gamma^2 + \beta}$, that is 1.414. Of course, such results suggest that the method is overly conservative, since the control input $u(t)$ is far in figure 3 from the saturation limit. Note that the control input $u(t)$ is guaranteed no saturating for all

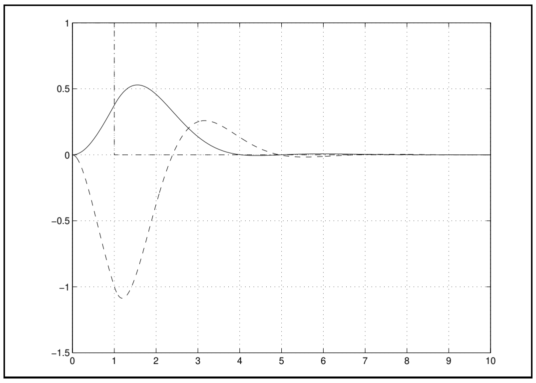

Figure 3: Time responses

initial condition $x(0) \in \mathcal{P}$ and all signal $w(t)$ whose energy is less than one. To adequately mesure the conservatism, it is necessary to compute the maximum modulus of the control input $u(t)$ for all $x(0) \in \mathcal{P}$ and for all signal $w(t)$ whose energy is less than one. Such a computation is outside the scope of this article.

8 Conclusion

In this paper, we considered the control of an LTI system with input saturations. Two approaches were considered. The first approach implies that the designed control law never saturates the actuator. The obtained controller is LTI. The second one ensures absolute stability against a portion of the nonlinearity. The obtained controller is LTV. As seen in the section 5, this approach can be extended to obtain an LTI controller, with a (possible) loss of performance.

A mere comparison of the obtained LMI formulation for both problems does not allow to compare the efficiency of the two approaches. Even if the *saturating controller* has a more complex structure than the *non saturating controller*, numerical experiments suggest that, for the considered problem, it does not perform better. Furthermore, with a *non saturating controller*, the closed loop system can be considered as LTI. If $\mathcal{L}_2$ gain stability ensures desirable properties to linear systems, it is no longer true when considering nonlinear one [11]. Consider, for instance, the steady state properties: it is easy to exhibit a nonlinear, $\mathcal{L}_2$ gain stable system whose response to a constant input does not converge to a constant [9, 11]. *Incremental* $\mathcal{L}_2$ stability ensures the steady state properties [10, 9, 11]. LTI $\mathcal{L}_2$ gain stable systems are *incrementally* stable too. As a consequence, the *non saturating controller*, in contrast with the *saturating controller*, guarantees the steady state properties (and many others, see [11] and references within).

Nevertheless, it is worth noting that the *saturating controller* ensures at the input of the system intrinsic margin stability properties [15].

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Appendix

A Proof of theorem 4.1

The proof is in two parts. First, assuming that the actuators do not saturate, we prove that conditions (4), (5) and (7) ensure that the system is $\{\mathcal{P}, \gamma, \beta\}$. The second part consists in proving that the conditions (6) and (7) ensure that the actuators does not actually saturate.

• Assume that the actuator does not saturate. The closed loop system has then the following equations:

$$\begin{bmatrix} \dot{\tilde{x}} \\ z \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_w \\ \tilde{C}_z & \tilde{D}_{zw} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}, \tag{13}$$

where $\tilde{x} = [x^T \; \bar{x}^T]^T$, with

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & \tilde{D}_{zw} \end{array}\right] = \left[\begin{array}{cc|c} A & B_u\bar{C} & B_w \\ \bar{B}C_y & \bar{A} & \bar{B}D_{yw} \\ \hline C_z & D_{zu}\bar{C} & D_{zw} \end{array}\right].$$

The closed-loop system is stable and has an $\mathcal{L}_2$-gain less than γ if there exists a quadratic function $V(x) = \tilde{x}^T \tilde{P} \tilde{x}$ such that, for all $T > 0$,

$$V(\tilde{x}(T)) + \int_0^T z(t)^T z(t) dt \leq V(\tilde{x}(0)) + \gamma^2 \int_0^T w(t)^T w(t) dt. \tag{14}$$

Thus, the closed loop system is $(\mathcal{P}, \gamma, \beta)$ if for any $\tilde{x}(0) \in \mathcal{P}$, $V(\tilde{x}(0)) \leq \beta$. This fact is ensured by:

$$\begin{bmatrix} v_j \\ 0 \end{bmatrix}^T \tilde{P} \begin{bmatrix} v_j \\ 0 \end{bmatrix} \leq \beta, \text{ for } j = 1, \ldots, r. \tag{15}$$

Let us focus on the condition (14). It is a well-known fact (bounded-real lemma) that for $V(x) = \tilde{x}^T \tilde{P} \tilde{x}$, the condition (14) is equivalent to:

$$\begin{bmatrix} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B}_w & \tilde{C}_z \\ \tilde{B}_w^T\tilde{P} & -\gamma^2 I & \tilde{D}_{zw} \\ \tilde{C}_z^T & \tilde{D}_{zw}^T & -I \end{bmatrix} < 0. \tag{16}$$

The condition (7) implies that

$$\begin{bmatrix} Q & \gamma I \\ \gamma I & P \end{bmatrix} > 0, \tag{17}$$

which implies that if the upper-left block of $\tilde{P}$ is P, and that of $\tilde{P}^{-1}$ is $\gamma^{-2}Q$ then $\tilde{P} > 0$, with $\tilde{P}$ (and $\tilde{P}^{-1}$) parameterized as follows. For an arbitrary invertible matrix $M \in \mathbf{R}^{n\times n}$:

$$\left\{ \begin{array}{lcl} \tilde{P} & = & \begin{bmatrix} I & 0 \\ 0 & M^T \end{bmatrix} \begin{bmatrix} P & I \\ I & (P-\gamma^2 Q^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \\ \tilde{P}^{-1} & = & \begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix} \begin{bmatrix} \gamma^{-2}Q & I \\ I & (\gamma^{-2}Q - P^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \end{array} \right. \tag{18}$$

where $N = (I - \gamma^{-2}QP)M^{-T}$. Let us introduce the variables Y and Z such that:

$$Y \;=\; \bar{C}M^{-1}(\gamma^2 I - PQ) = \gamma^2 \bar{C} N^T \qquad \text{and} \qquad Z \;=\; M\bar{B}. \tag{19}$$

We seek matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ such that the condition (16) is satisfied. By a direct application of the Schur lemma [16, page 472], condition (16) is equivalent to:

$$\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix} < 0, \tag{20}$$

and

$$F = E_{11} - \begin{bmatrix} E_{12} & E_{13} \end{bmatrix} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix}^{-1} \begin{bmatrix} E_{21} \\ E_{31} \end{bmatrix} < 0,$$

where $E_{11}, \ldots, E_{33}$ are the sub blocks of the matrix (16). Let us introduce the notation:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}.$$

We now introduce a technical lemma, from [21].

Lemma A.1 *Let be the matrices $F \in \mathbf{R}^{\mathbf{2n}\times\mathbf{2n}}$, $\tilde{F} \in \mathbf{R}^{\mathbf{2n}\times\mathbf{2n}}$, $T \in \mathbf{R}^{\mathbf{2n}\times\mathbf{2n}}$ where:*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}, \quad \hat{F} = \hat{T}^T F \hat{T} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ \hat{F}_{12}^T & \hat{F}_{22} \end{bmatrix}, \quad \hat{T} = \begin{bmatrix} I & 0 \\ -R^T & I \end{bmatrix},$$

with the invertible matrix $R \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$. If the following conditions hold: $F_{11} > 0$, $\hat{F}_{11} > 0$ and $\hat{F}_{12} = -\hat{F}_{11}R^{-T}$ then the matrix F is positive definite.

Proof The result derives from the proof of the Schur lemma [16, page 472]. □

By applying lemma A.1 with $R = M$ and after some cumbersome manipulations, condition $F_{11} < 0$ is equivalent to:

$$\left[\begin{array}{c|c} \begin{array}{c} A^TP + PA + C_y^TZ^T \\ +ZC_y + C_z^TC_z \end{array} & \begin{array}{c} PB_w + ZD_{yw} \\ +C_z^TD_{zw} \end{array} \\ \hline \begin{array}{c} B_w^TP + D_{yw}^TZ^T \\ +D_{zw}^TC_z \end{array} & D_{zw}^TD_{zw} - \gamma^2 I \end{array} \right] < 0. \tag{21}$$

Condition $\hat{F}_{11} < 0$ can be written as:

$$\left[\begin{array}{c|c} AQ + QA^T + B_uY & QC_z^T + Y^T D_{zu}^T \\ +Y^T B_u^T + B_w B_w^T & +B_w D_{zw}^T \\ \hline C_zQ + D_{zu}Y & \\ +D_{zw}B_w^T & D_{zw}D_{zw}^T - \gamma^2 I \end{array}\right] < 0. \tag{22}$$

Assume that P, Q, Y, Z, S satisfy the inequalities (21) and (22). To satisfy condition $\hat{F}_{12} = -\hat{F}_{11}R^{-T}$, $\bar{A}$ is selected such that:

$$\begin{aligned} \bar{A} = M^{-1}\Bigg(& PA + \gamma^2 A^T Q^{-1} + PB_uYQ^{-1} + ZC_y \\ - & \begin{bmatrix} (PB_w + ZD_{yw})^T \\ C_z \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I & D_{zw}^T \\ D_{zw} & -I \end{bmatrix}^{-1} \begin{bmatrix} \gamma^2 B_w^T Q^{-1} \\ C_z + D_{zu}YQ^{-1} \end{bmatrix}\Bigg) M^{-1}. \end{aligned} \tag{23}$$

From the parameterization of $\tilde{P}$, condition (15) leads to: $v_j^T P v_j \leq \beta$, for $j = 1, \ldots, r$.

• We now prove that the conditions (6) and (7) ensures that the actuators does not saturate. From the condition (14), we deduce that for any $T \geq 0$:

$$V(x(T)) \leq V(\tilde{x}(0)) + \gamma^2 \int_0^T w(t)^T w(t) dt \leq \beta + \gamma^2.$$

that is, for any T, $\tilde{x}(T)$ always belongs to the ellipsoid: $\mathcal{E}_{\frac{1}{\beta+\gamma^2}\tilde{P}} = \left\{\zeta \in \mathbf{R}^{2n} \mid \zeta^T \tilde{P} \zeta \leq \beta + \gamma^2\right\}$. The actuators do not saturate if $u_i(t)^2 \leq u_{i_{sat}}^2$, that is, as $u_i(t) = \tilde{C}_{u_i}\tilde{x}(t)$, with $\tilde{C}_{u_i} = \begin{bmatrix} 0 & \bar{C}_i \end{bmatrix}$ where $\bar{C}_i$ is the i^{th} row of the matrix $\bar{C}$, we have: $\tilde{C}_{u_i}\tilde{P}^{-1}\tilde{C}_{u_i}^T \leq \frac{1}{\beta+\gamma^2}u_{i_{\rm sat}}^2$, for $i = 1, \ldots, n_u$. that is, $\bar{C}_i N^T(\gamma^{-2}Q - P^{-1})^{-1} N \bar{C}_i^T \leq \frac{1}{\beta+\gamma^2}u_{i_{\rm sat}}^2$, which is equivalent to:

$$u_{i_{\rm sat}} \begin{bmatrix} \frac{\gamma^2}{\gamma^2+\beta} I & 0 & 0 \\ 0 & Q & \gamma I \\ 0 & \gamma I & P \end{bmatrix} + \begin{bmatrix} 0 & Y_i & 0 \\ Y_i^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0, \quad \text{for } i = 1, \ldots, n_u \tag{24}$$

where Y_i is the i^{th} row of Y.

B Proof of theorem 4.2

The theorem 4.2 can be proved from the results of [32]. For the sake of completeness, an alternative proof is proposed here, with a different way to get an explicit expression for the control law. In contrast with [32], once the existence of the control law is proved, the control law is obtained through an explicit expression. This proof is similar to the first part of the previous one, even if it is a little bit complicated. In the section 3, the LTI system with input saturations was interpreted as a Linear Parameter Dependent (LPV) system [24, 1, 31, 32], as it can be rewritten as:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left(\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \\ 0 \end{bmatrix} \Delta(t) \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix}^T \right) \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{25}$$

where the time-varying gain $\Delta(t)$ is defined as:

$$\Delta(t) = \begin{bmatrix} \delta_1(t) & 0 & \dots & \dots & 0 \\ 0 & \delta_2(t) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \dots & 0 & \delta_{n_u}(t) \end{bmatrix}$$

with $\delta_i(t) = 1 - \frac{\mathbf{sat}(u_i(t))}{u_i(t)}$. As it was discussed in the section 3, we have $0 \leq \delta_i \leq \sigma_i = 1 - u_{i\mathrm{sat}}/u_{i\max} < 1$. For this class of system, we seek a controller with the following structure:

$$\begin{bmatrix} \dot{\bar{x}} \\ u \end{bmatrix} = \left(\begin{bmatrix} \bar{A} & \bar{B}_y \\ \bar{C}_u & 0 \end{bmatrix} + \begin{bmatrix} \bar{B}_{\bar{p}} \\ 0 \end{bmatrix} \Delta(t) \left(I - \bar{D}_{\bar{q}\bar{p}} \Delta(t)\right)^{-1} \begin{bmatrix} \bar{C}_{\bar{q}} & \bar{D}_{\bar{q}y} \end{bmatrix} \right) \begin{bmatrix} \bar{x} \\ y \end{bmatrix}, \tag{26}$$

which corresponds to the controller (3). $\bar{x} : \mathbf{R}_+ \to \mathbf{R}^n$ is the state of the control law and $\bar{A}$, $\bar{B}_y$, $\bar{B}_{\bar{p}}$, $\bar{C}_u$, $\bar{C}_{\bar{q}}$, $\bar{D}_{\bar{q}\bar{p}}$, $\bar{D}_{\bar{q}y}$ are real matrices to find.

Using this modeling, the closed-loop system has the following equations, with $\tilde{x} = \begin{bmatrix} x^T & \bar{x}^T \end{bmatrix}^T$:

$$\begin{bmatrix} \dot{\tilde{x}} \\ z \end{bmatrix} = \begin{bmatrix} \tilde{A}(\tilde{\Delta}) & \tilde{B}_w(\tilde{\Delta}) \\ \tilde{C}_z(\tilde{\Delta}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}, \tag{27}$$

where

$$\begin{bmatrix} \tilde{A}(\tilde{\Delta}) & \tilde{B}_w(\tilde{\Delta}) \\ \tilde{C}_z(\tilde{\Delta}) & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_w \\ \tilde{C}_z & 0 \end{bmatrix} + \begin{bmatrix} \tilde{B}_{\tilde{p}} \\ \tilde{D}_{z\tilde{p}} \end{bmatrix} \tilde{\Delta} \left(I - \tilde{D}_{\tilde{q}\tilde{p}} \tilde{\Delta}\right)^{-1} \begin{bmatrix} \tilde{C}_{\tilde{q}}^T \\ \tilde{D}_{\tilde{q}w} \end{bmatrix},$$

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_{\tilde{p}} & \tilde{B}_w \\ \hline \tilde{C}_{\tilde{q}} & \tilde{D}_{\tilde{q}\tilde{p}} & \tilde{D}_{\tilde{q}w} \\ \hline \tilde{C}_z & \tilde{D}_{z\tilde{p}} & \tilde{D}_{zw} \end{array}\right] = \left[\begin{array}{cc|cc|c} A & B_u\bar{C}_u & B_u & 0 & B_w \\ \bar{B}_y C_y & \bar{A} & 0 & \bar{B}_{\bar{p}} & \bar{B}_y D_{yw} \\ \hline 0 & -\bar{C}_u & 0 & 0 & 0 \\ \bar{D}_{\bar{q}y} C_y & \bar{C}_{\bar{q}} & 0 & \bar{D}_{\bar{q}\bar{p}} & \bar{D}_{\bar{q}y} D_{yw} \\ \hline C_z & D_{zu}\bar{C}_u & -D_{zu} & 0 & D_{zw} \end{array}\right],$$

with $\tilde{\Delta} = \begin{bmatrix} \Delta(t) & 0 \\ 0 & \Delta(t) \end{bmatrix}$.

From [32], the closed-loop system defined by (27) is stable and has an $\mathcal{L}_2$-gain less than γ between the input w and the output z if there exist two positive definite matrices, $\tilde{P} \in \mathbf{R}^{2n \times 2n}$ and $\tilde{S} = \begin{bmatrix} S & S_{12} \\ S_{12} & S_{22} \end{bmatrix}$ with $S, S_{12}, S_{22} \in \mathcal{S}(n_u)$ such that:

$$\left[\begin{array}{c|c|c|c} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & P\tilde{B}_{\tilde{p}} + \tilde{C}_{\tilde{q}}^T\tilde{V}\tilde{S} & \tilde{P}\tilde{B}_w & \tilde{C}_z^T \\ \hline \tilde{B}_{\tilde{p}}^T P + \tilde{V}\tilde{S}C_{\tilde{q}} & \begin{array}{c}\tilde{V}\tilde{S}\tilde{D}_{\tilde{q}\tilde{p}} + \tilde{D}_{\tilde{q}\tilde{p}}^T\tilde{V}\tilde{S} \\ -2\tilde{S}\end{array} & \tilde{V}\tilde{S}D_{\tilde{q}w} & \tilde{D}_{z\tilde{p}}^T \\ \hline \tilde{B}_w^T\tilde{P} & \tilde{D}_{\tilde{q}w}^T\tilde{V}\tilde{S} & -\gamma^2 I & \tilde{D}_{zw}^T \\ \hline \tilde{C}_z & \tilde{D}_{z\tilde{p}} & \tilde{D}_{zw} & -I \end{array}\right] < 0\,, \tag{28}$$

with $\tilde{V} = \mathbf{diag}\ (\sigma_{\mathbf{1}}, ..., \sigma_{\mathbf{n_u}}, \sigma_{\mathbf{1}}, ..., \sigma_{\mathbf{n_u}})$. This inequality ensures that the closed-loop system stability is proved through the use of the Lyapunov function $V(\tilde{x}) = \tilde{x}\tilde{P}\tilde{x}$. As in the section A, the closed-loop system is $\{\mathcal{P}, \gamma, \beta\}$ if, in addition to the inequality (28), we have:

$$\begin{bmatrix} v_j \\ 0 \end{bmatrix}^T \tilde{P} \begin{bmatrix} v_j \\ 0 \end{bmatrix} \leq \beta, \qquad \text{for } j = 1, \ldots, r. \tag{29}$$

In the same way, the following inequality ensures that the i^{th} control input is bounded by $u_{i_{max}}$:

$$\tilde{C}_{u_i}\tilde{P}^{-1}\tilde{C}_{u_i}^T \leq \frac{1}{\beta + \gamma^2} u_{i\,\max}^2, \qquad \text{for } i = 1, \ldots, n_u. \tag{30}$$

As in the section A, the condition (7) implies that $\begin{bmatrix} Q & \gamma I \\ \gamma I & P \end{bmatrix} > 0$, which implies that if the upper-left block of $\tilde{P}$ is P, and that of $\tilde{P}^{-1}$ is $\gamma^{-2}Q$ then $\tilde{P} > 0$, with $\tilde{P}$ parameterized as follows. For an arbitrary invertible matrix $M_1 \in \mathbf{R}^{n \times n}$:

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & M_1^T \end{bmatrix} \begin{bmatrix} P & I \\ I & (P - \gamma^2 Q^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M_1 \end{bmatrix} \tag{31}$$

where $N_1 = (I - \gamma^{-2}QP)M_1^{-T}$. The condition (10) ensures that if the upper-left block of $\tilde{S}$ is S, and that of $\tilde{S}^{-1}$ is $\gamma^{-2}T$ then $\tilde{S} > 0$, with $\tilde{S}$ parameterized as follows. For an arbitrary invertible matrix $M_2 \in \mathcal{S}(n_u)$, with $N_2 = (I - \gamma^{-2}ST)M_2^{-T}$:

$$\tilde{S} = \begin{bmatrix} I & 0 \\ 0 & M_2^T \end{bmatrix} \begin{bmatrix} S & I \\ I & (S - \gamma^2 T^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M_2 \end{bmatrix}. \tag{32}$$

Let us introduce the variables Y and Z such that:

$$Y = \bar{C}_u M_1^{-1}(\gamma^2 I - PQ), \quad Z_1 = M_1 \bar{B}_y, \quad Z_2 = M_2 \bar{D}_{\bar{q}y}. \tag{33}$$

We seek matrices $\bar{A}$, $\bar{B}_y$, $\bar{B}_{\bar{p}}$, $\bar{C}_u$, $\bar{C}_{\bar{q}}$, $\bar{D}_{\bar{q}\bar{p}}$, $\bar{D}_{\bar{q}y}$ such that the condition (28) is satisfied. By a direct application of the Schur lemma [16, page 472], condition (16) is equivalent to:

$$\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix} < 0, \tag{34}$$

and

$$F = E_{11} - \begin{bmatrix} E_{12} & E_{13} \end{bmatrix} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix}^{-1} \begin{bmatrix} E_{21} \\ E_{31} \end{bmatrix} < 0,$$

where $E_{11}, \cdots, E_{33}$ are the sub blocks of the matrix (28). Let us introduce the notation:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}.$$

By applying the lemma A.1 and after some cumbersome manipulations, the inequality $F_{11} < 0$ gives:

$$\left[\begin{array}{c|c|c} \begin{array}{c} A^T P + C_y^T Z_1^T \\ +PA + Z_1 C_y \\ +C_z^T C_z \end{array} & \begin{array}{c} PB_u + C_y^T Z_2^T V \\ +C_z^T D_{zu} \end{array} & \begin{array}{c} PB_w + Z_1 D_{yw} \\ +C_z^T D_{zw} \end{array} \\ \hline \begin{array}{c} B_u^T P + V Z_2 C_y \\ +D_{zu}^T C_z \end{array} & D_{zu}^T D_{zu} - 2S & \begin{array}{c} V Z_2 D_{yw} \\ +D_{zu}^T D_{zw} \end{array} \\ \hline \begin{array}{c} B_w^T P + +D_{yw}^T Z_1^T \\ D_{zw}^T C_z \end{array} & \begin{array}{c} D_{yw}^T Z_2^T V \\ +D_{zw}^T D_{zu} \end{array} & D_{zw}^T D_{zw} - \gamma^2 I \end{array}\right] < 0. \tag{35}$$

With $R = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, $M_1 M_1^T = P - \gamma^2 Q^{-1}$, and $M_2 M_2^T = S - \gamma^2 T^{-1}$, the condition $\hat{F}_{11} < 0$ leads to:

$$\left[\begin{array}{c|c|c} \begin{array}{c} QA^T + Y^T B_u^T \\ +AQ + B_u Y \\ +B_w B_w^T \end{array} & B_u T - Y^T V & \begin{array}{c} QC_z^T + Y^T D_{zu}^T \\ +B_w D_{zw}^T \end{array} \\ \hline TB_u^T - VY & -2T & TD_{zu}^T \\ \hline \begin{array}{c} C_z Q + D_{zu} Y \\ +D_{zw} B_w^T \end{array} & D_{zu} T & D_{zw} D_{zw}^T - \gamma^2 I \end{array}\right] < 0. \tag{36}$$

For P, Q, Y, Z, S satisfying the inequalities (35) and (36), the condition $\hat{F}_{12} = -\hat{F}_{11}R^{-T}$ is satisfied, by choosing $\bar{A}$, $\bar{B}_{\bar{p}}$, $\bar{C}_{\bar{q}}$, $\bar{D}_{\bar{q}\bar{p}}$ such that:

$$\begin{bmatrix} \bar{A} & \bar{B}_{\bar{p}} \\ \bar{C}_{\bar{q}} & \bar{D}_{\bar{q}\bar{p}} \end{bmatrix} = \begin{bmatrix} M_1^{-1} & 0 \\ 0 & V^{-1}M_2^{-1} \end{bmatrix} \left(\left(\begin{bmatrix} PA + \gamma^2 A^T Q^{-1} \\ +PB_uYQ^{-1} + Z_1C_y \\ \gamma^2 B_u^T Q^{-1} + VZ_2C_y \\ -VSYQ^{-1} \end{bmatrix} \cdots \right.\right.$$

$$\cdots \left. \begin{matrix} PB_u \\ 2\gamma^2 T^{-1} \end{matrix} \right] - \begin{bmatrix} PB_w \\ +Z_1D_{yw}^T & C_z^T \\ Z_2D_{yw} & D_{zu}^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & D_{zw}^T \\ D_{zw} & -I \end{bmatrix}^{-1} \cdots \tag{37}$$

$$\cdots \left. \left. \begin{bmatrix} \gamma^2 B_w^T Q^{-1} & 0 \\ C_z \\ +D_{zu}YQ^{-1} & D_{zu} \end{bmatrix} \right) \right) \begin{bmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{bmatrix}.$$

From conditions (33), $\bar{B}_y = M_1^{-1}Z_1$, $\bar{D}_{\bar{q}y} = M_2^{-1}Z_2$, $\bar{C}_u = -YQ^{-1}M_1^{-1}$. The conditions (29) and (30) are equivalent to: $v_j^T P v_j \leq \beta, \quad \text{for } j = 1, \ldots, r \quad \text{and}$

$$u_{i\max} \begin{bmatrix} \frac{\gamma^2}{\gamma^2+\beta} I & 0 & 0 \\ 0 & Q & \gamma I \\ 0 & \gamma I & P \end{bmatrix} + \begin{bmatrix} 0 & Y_i & 0 \\ Y_i^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, n_u, \tag{38}$$

where Y_i is the i^{th} row of Y.

C Proof of theorem 5.1

The proof is similar to the proof of theorem 4.2. The system is modeled as the LPV system (25). In contrast with the previous case, we want to design an *LTI* controller. As a consequence, the proof developed in the section B is modified with $\tilde{S} = S \in \mathcal{S}(n_u)$ and $\tilde{S}^{-1} = T$.