EECS 227BT 9/3/19
L. El Ghaoui, S. Sojoudi

Homework Assignment #1

Due date: 9/12/19, before class. Please LaTeX your homework solution and submit an electronic version.

Exercise 1 (About general optimization) In this exercise, we test your understanding of the general framework of optimization and its language. We consider an optimization problem in standard form:

$$p^* = \min_{x \in \mathbb{R}^n} f_0(x) \ : \ f_i(x) \leq 0, \ \ i = 1, \ldots, m.$$

In the following we denote by $\mathcal{X}$ the feasible set. For the following statements, provide a proof or counter-example.

1. Any optimization problem can be expressed as one with a linear objective.
2. Any optimization problem can be expressed as one without any constraints.
3. Any optimization problem can be recast as a linear program, provided one allows for an infinite number of constraints.
4. If there are strict inequalities we can always replace them with non-strict ones. *Note:* you are not required to provide a formal justification.
5. If the problem involves the minimization of an objective function of the form

$$f_0(x) = \min_y F_0(x, y),$$

then it is possible to exchange the minimization over y and x, without altering the optimal value.

6. If the problem involves the minimization of an objective function of the form

$$f_0(x) = \max_y F_0(x, y),$$

then $p^* \geq d^*$, where

$$d^* := \max_y \min_{x \in \mathcal{X}} F_0(x, y),$$

where $\mathcal{X}$ is the feasible set of the original problem. *Hint:* consider the function $y \to \min_{x'} F_0(x', y)$ and a similar function of x.

Exercise 2 (Recognize convex sets) In this exercise, we test your understanding of convex sets. For the following statements, prove whether the given set is convex or not.

1. A rectangle, *i.e.*a set of the form $\{x \in \mathbf{R}^n | \alpha_i \le x_i \le \beta_i, \ \forall i \in \{1, \dots, n\}$.
2. The set of points closer to a given point than to a given set, *i.e.*, for $S \subset \mathbf{R}^n$,
$$\{x | \|x - x_0\|_2 \le \|x - y\|_2, \ \forall y \in S\} \tag{1}$$
3. The set
$$\{x | x + S_2 \subset S_1\} \tag{2}$$
where $S_1, \ S_2 \subset \mathbf{R}^n$ and S_1 is convex.
4. The set of points below all the graphs of a family of convex functions, *i.e.* for convex $(g_i)_{i \in S} : \mathbf{R}^n \mapsto \mathbf{R}$,
$$\{(x, t) \in \mathbf{R}^n \times \mathbf{R}, t \le g_i(x) \ \forall i \in S\} \tag{3}$$

Exercise 3 (Convex functions) In this exercise, we test your understanding of convex functions. For the following functions, prove whether it is convex, concave or neither.

1. $f : (x_1, x_2) \mapsto x_1 x_2$
2. $f : (x_1, x_2) \mapsto -\frac{1}{x_1 x_2}$ on $\mathbf{dom}\, f = \mathbf{R}^2_{++}$
3. $LSE : x \in \mathbf{R}^n \mapsto \log(\sum_{i=1}^n \exp(x_i))$
4. $f : (X, y) \mapsto y^T X^{-1} y$ on $\mathbf{dom}\, f = \mathbb{S}^n_{++} \times \mathbf{R}^n$

Exercise 4 (Properties of convex functions) In this exercise, we use convexity of particular functions to deduce useful inequalities.

1. Show that for all $x \in [0, \frac{\pi}{2}]$,
$$\frac{2}{\pi} x \le \sin x \le x \tag{4}$$
2. Show that for all $a, \ b \in \mathbf{R}^n_{++}$,
$$\left(\prod_{i=1}^n a_i\right)^{1/n} + \left(\prod_{i=1}^n b_i\right)^{1/n} \le \left(\prod_{i=1}^n (a_i + b_i)\right)^{1/n} \tag{5}$$

Exercise 5 (Properties of positive semi-definite matrices) Let A, B be two of symmetric positive semidefinite $n \times n$ matrices. Prove the following, or provide a counter-example.

1. $2A + 3B$ is positive semi-definite.

2. $\mathbf{Tr}\, AB \geq 0$.

3. $\mathbf{Tr}\, AB = 0$ if and only if $AB = 0$.

4. The matrix C with elements given by $C_{ij} = A_{ij}B_{ij}$, $1 \leq i, j \leq n$, is positive semi-definite.

5. If $A - B$ is positive semi-definite, then $\lambda_{\max}(A) \geq \lambda_{\max}(B)$, where $\lambda_{\max}$ denotes the largest eigenvalue.

6. If $A - B$ is positive semi-definite, then for every $k = 1, \ldots, n$, $\lambda_k(A) \geq \lambda_k(B)$, where λ_k denotes the k-th largest eigenvalue. *Hint:* you can use without proof the following "variational" (that is, optimization-based) characterization of the k-th largest eigenvalue of a symmetric matrix:
$$\lambda_k(A) = \min_V \max_{x \in V,\, \|x\|_2 = 1} x^T A x.$$
where the minimum is taken with respect to all subpsaces of dimension $n - k + 1$.

Exercise 6 (Reformulating constaints in `cvx`) Each of the following `cvx` code fragments describes a convex constraint on the scalar variables `x`, `y`, and `z`, but violates the `cvx` rule set, and so is invalid. Briefly explain why each fragment is invalid. Then, rewrite each one in an equivalent form that conforms to the `cvx` rule set. In your reformulations, you can use linear equality and inequality constraints, and inequalities constructed using `cvx` functions. You can also introduce additional variables, or use LMIs. Be sure to explain (briefly) why your reformulation is equivalent to the original constraint, if it is not obvious.

Check your reformulations by creating a small problem that includes these constraints, and solving it using `cvx`. Your test problem doesn't have to be feasible; it's enough to verify that `cvx` processes your constraints without error.

Remark. This *looks* like a problem about 'how to use `cvx` software', or 'tricks for using `cvx`'. But it really checks whether you understand the various composition rules, convex analysis, and constraint reformulation rules.

1. `norm( [ x + 2*y , x - y ] ) == 0`

2. `square( square( x + y ) ) <= x - y`

3. `1/x + 1/y <= 1; x >= 0; y >= 0`

4. `norm([ max( x , 1 ) , max( y , 2 ) ]) <= 3*x + y`

5. `x*y >= 1; x >= 0; y >= 0`

6. `( x + y )^2 / sqrt( y ) <= x - y + 5`

7. `x^3 + y^3 <= 1; x>=0; y>=0`

8. `x+z <= 1+sqrt(x*y-z^2); x>=0; y>=0`