EE227BT Discussion Section #8

Exercise 1 (Sum Of Squares) Let $\mathbb{R}[x]_{2d} := \{p \in \mathbb{R}[x] \mid \deg(p) \leq 2d\}$ be the vector space of univariate polynomials of degree at most $2d$.

Let $P_{2d} := \{p \in \mathbb{R}[x]_{2d} \mid p(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$ be the set of univariate polynomials of degree at most $2d$ which are *nonnegative*.

Let $\Sigma_{2d} := \{p \in \mathbb{R}[x]_{2d} \mid p(x) = \sum_{i=1}^{m} q_i(x)^2,\ m \in \mathbb{N},\ q_i \in \mathbb{R}[x]_d\}$ be the set of univariate polynomials of degree at most $2d$ which can be written as a *sum of squares*.

1. Show that

$$\Sigma_{2d} = \left\{ p = \begin{bmatrix} c_0 & \dots & c_{2d} \end{bmatrix}^\top \in \mathbb{R}[x]_{2d} \;\middle|\; c_i = \sum_{k+l=i} Y_{kl},\ i = 0, \dots, 2d, \text{ for some } Y \in \mathbb{S}_+^{d+1} \right\}$$

2. Prove that $P_{2d} = \Sigma_{2d}$.

3. Let $p = \begin{bmatrix} c_0 & c_1 & \dots & c_{2d} \end{bmatrix}^\top \in \mathbb{R}[x]_{2d}$. Formulate the optimization problem $\min_{x \in \mathbb{R}} p(x)$ as a semidefinite program.

4. Show that P_{2d} is a proper cone.

5. Show that $P_{2d}^* = K_{\text{han}}$, where

$$K_{\text{han}} := \{\mu \in \mathbb{R}^{2d+1} \mid H(\mu) \succeq 0\}$$

and

$$H(\mu) := \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{d-1} & \mu_d \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_d & \mu_{d+1} \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{d+1} & \mu_{d+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{d-1} & \mu_d & \mu_{d+1} & \cdots & \mu_{2d-2} & \mu_{2d-1} \\ \mu_d & \mu_{d+1} & \mu_{d+2} & \cdots & \mu_{2d-1} & \mu_{2d} \end{bmatrix}$$