

IMA 2003 Workshop, March 12-19, 2003

A Robust Option Pricing Problem

Laurent El Ghaoui

Department of EECS, UC Berkeley

Robust optimization

standard form:

$$\min_x \sup_{u \in \mathcal{U}} f_0(x, u) : \forall u \in \mathcal{U}, f_i(x, u) \leq 0, \quad i = 1, \dots, m$$

- $x \in \mathbf{R}^n$ is the decision variable
- u is a parameter vector affecting the problem data
- set \mathcal{U} describes uncertainty on u

a **semi-infinite** optimization problem

Agenda

- option basics
- option pricing problem
- worst-case approach

joint work with: A. d'Aspremont

Options

Options are time bombs. —Warren Buffett, 2003

let $x(T)$ denote the price of an asset at time T

an *European call option* with maturity T and strike price K is a contract with payoff

$$(x(T) - K)_+ = \sup(x(T) - K, 0)$$

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*what is the price of the option,
assuming no arbitrage ("free lunch") is possible?*

Price of option

fundamental result of finance: assuming a zero risk-free interest rate, the no-arbitrage price of the option is

$$\mathbf{E}_\pi(x(T) - K)_+$$

for some distribution π on the asset price at time T

such a π is called **risk-neutral**

Forwards

under the risk-neutral distribution π , the **current** asset price is

$$\mathbf{E}_{\pi} x(T)$$

i.e., π is a martingale

Basket options

let $x(T)$ be a n -vector of prices of assets at time T

a **basket** option with weight vector w and strike K is the contract with payoff

$$(w^T x(T) - K)_+$$

denote the basket option by (w, K) and its price by

$$C_\pi(w, K) := \mathbf{E}_\pi(w^T x(T) - K)_+$$

(maturity date is implicit here)

Option pricing problem

given

- w_0, w_1, \dots, w_m in \mathbf{R}_+^n (*basket weights*)
- K_0, K_1, \dots, K_m in \mathbf{R}_+ (*strike prices*)
- p_1, \dots, p_m in \mathbf{R}_+ (*observed option prices*)

determine the price of the basket option with weight w_0 and strike price K_0

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in practice, we are also given the **current** asset prices themselves,
 $q \in \mathbf{R}_+^n$ ("forwards")

Challenges & methods

challenges:

- the risk-neutral measure is **not** the empirical distribution of assets
- hence, we may have to rely on option & forward prices only

two approaches:

- model-based approach
- arbitrage-based approach

Model-based approach

- assume a log-normal diffusion model for the asset prices

$$ds = Asdt + Bsdw,$$

where $s = \log x$, and w is a multidimensional Brownian motion

- fit the model to observed option prices

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with some approximations, this problem is an SDP (Aspremont, 2000)

Model-based approach

- assume a log-normal **diffusion model** for the asset prices

$$ds = Asdt + Bsdw,$$

where $s = \log x$, and w is a multidimensional Brownian motion

- fit the model to observed option prices

pros and cons:

- very versatile, and "easy" to solve (albeit only recently)
- makes a **structural assumption** about the risk-neutral measure π
- provides a point estimate for the price of basket (w_0, K_0)

No-arbitrage approach

find **bounds** on the price of basket (w_0, K_0) by solving semi-infinite LP

$$\begin{aligned} & \sup_{\pi} / \inf_{\pi} \quad \mathbf{E}_{\pi}(w_0^T x - K_0)_+ \\ & \text{s.t.} \quad \mathbf{E}_{\pi}(w_i^T x - K_i)_+ = p_i, \quad i = 1, \dots, m \end{aligned}$$

the optimization variable is the risk-neutral probability measure π

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pros and cons:

- problem may be **difficult**
- approach provides only bounds, but ...
- ... makes **no assumptions** about market dynamics

Upper bound problem

upper bound problem:

$$p^{\text{sup}} := \sup_{\pi \in \mathcal{P}} \int_{\Omega} \phi_0(x) \pi(x) dx \quad : \quad \int_{\Omega} \pi(x) dx = 1, \\ \int_{\Omega} \phi_i(x) \pi(x) dx = p_i, \quad i = 1, \dots, m,$$

where

- $\Omega = \mathbf{R}_+^n$
- \mathcal{P} is the set of densities with support in Ω
- $\phi_i(x) := (w_i^T x - K_i)_+, \quad i = 0, 1, \dots, m$

Upper bound problem

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Lagrangian:

$$\mathcal{L}(\pi, \lambda, \lambda_0) = \int_{\Omega} \phi_0(x) \pi(x) dx + \lambda_0 \left(1 - \int_{\Omega} \pi(x) dx \right) \\ + \lambda^T \left(p - \int_{\Omega} \phi(x) \pi(x) dx \right)$$

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the dual is the [robust linear programming](#) problem

$$d^{\text{sup}} := \inf_{\lambda \in \mathbf{R}^m} \lambda^T p + \lambda_0 \\ \text{s.t.} \quad \forall x \in \Omega, \quad \lambda^T \phi(x) + \lambda_0 \geq \phi_0(x)$$

Dual gives a hedging strategy

let λ_0, λ be feasible for the dual:

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strategy: invest λ_i in basket (w_i, K_i) , λ_0 in cash

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strategy: invest λ_i in basket (w_i, K_i) , λ_0 in cash

- price of strategy: $\lambda^T p + \lambda_0$
- taking expectations in $(*)$, get

$$\lambda^T p + \lambda_0 \geq \mathbf{E}_\pi \phi_0(x) = p^{\text{sup}}$$

(proves weak duality)

A special case

we are given option prices on the $m = n$ individual assets, as well as forward prices

$$\begin{aligned} \min_{\pi} / \sup_{\pi} \quad & \mathbf{E}_{\pi}(w_0^T x - K_0)_+ \\ \text{s.t.} \quad & \mathbf{E}_{\pi}(x_i - K_i)_+ = p_i, \quad i = 1, \dots, n \\ & \mathbf{E}_{\pi}x = q \end{aligned}$$

we may further relax the problem by ignoring the forward price information

Special case : upper bound

- **no-arbitrage**: problem is feasible iff $0 \leq p \leq q \leq p + K$
- upper bound is given by

$$d^{\text{sup}} = \max_{0 \leq j \leq n+1} w_0^T p + \sum_i w_{0,i} \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

where $\beta_0 = 0 \leq \beta_j := (q_j - p_j)/K_j \leq 1 = \beta_{n+1}$

- upper bound is attained, hence $d^{\text{sup}} = p^{\text{sup}}$

Ignoring forward prices

ignore the constraints

$$\mathbf{E}_\pi x = q$$

obtain formula by maximizing d^{sup} wrt q :

$$d^{\text{sup}} = p^{\text{sup}} = w_0^T p + (w_0^T K - K_0)_+$$

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makes [sense](#):

- concave in p (the RHS of our primal LP)
- convex in (w_0, K_0)
- interpolates p_i at $w_0 = i$ -th unit vector of \mathbf{R}^n , and $K_0 = K_i$

Sketch of proof

weak duality follows from homogeneity and convexity of $x \rightarrow x_+$:

$$\begin{aligned}\mathbf{E}_\pi(w_0^T x - K_0)_+ &= \mathbf{E}_\pi(w_0^T (x - K) + (w_0^T K - K_0))_+ \\ &\leq w_0^T \mathbf{E}_\pi(x - K)_+ + (w_0^T K - K_0)_+ \quad (w_0 \geq 0) \\ &= w_0^T p + (w_0^T K - K_0)_+\end{aligned}$$

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strong duality: choose $x = p + K$ with probability 1 if $w_0^T K \geq K_0$, otherwise take a limit of feasible distributions

$$x = \begin{cases} \epsilon^{-1} p + K & \text{with probability } \epsilon, \\ 0 & \text{with probability } 1 - \epsilon. \end{cases}$$

Lower bound

- dual problem reduces to a **finite LP**
(with $O(n)$ variables and constraints)

Lower bound

- dual problem reduces to a **finite LP**
(with $O(n)$ variables and constraints)
- if we ignore forward prices

$$p^{\text{inf}} = d^{\text{inf}} = \sum_{i : K_i w_i \geq K_0} p_i w_i + \max_{j : K_j w_j < K_0} \left(\sum_{i : K_i w_i < K_0} p_i w_i \min\left(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}\right) - K_0 + w_j K_j \right)_+$$

in which case, direct proof of perfect duality

General case: integral transform approach

the option price function

$$C(w, K) = \mathbf{E}_\pi(w^T x - K)_+$$

is an **integral transform** of measure π , with kernel the payoff function $(w^T x - K)_+$

General case: integral transform approach

the option price function

$$C(w, K) = \mathbf{E}_\pi(w^T x - K)_+ \quad (1)$$

is an **integral transform** of measure π , with kernel the payoff function $(w^T x - K)_+$

make C the variable, and solve **interpolation problem**

$$\sup_C / \inf_C C(w_0, K_0) : C(w_i, K_i) = p_i, i = 1, \dots, m, \\ C \text{ of the form (1)}$$

LP relaxation

conditions under which C is of form

$$C(w, K) = \mathbf{E}_\pi(w^T x - K)_+$$

for some measure π exist, but seem hard to check

semi-infinite LP relaxation:

$$\sup_C / \inf_C \quad C(w_0, K_0)$$

$$\text{s.t.} \quad C(w, K) \text{ convex in } (K, w)$$

$$C(w, K) \text{ homogeneous of degree 1}$$

$$-1 \leq \partial C(w, K) / \partial K \leq 0 \text{ and } C(w, K) \text{ nondecreasing in } w$$

$$C(w_i, K_i) = p_i, \quad i = 1, \dots, m$$

Finite LP formulation

optimal C of semi-infinite LP is **piecewise affine**

hence the semi-infinite LP can be reduced exactly to a finite LP

sup / inf p_0

subject to $\langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i, \quad i, j = 0, \dots, m + n + 1$

$g_{i,j} \geq 0, -1 \leq g_{i,n+1} \leq 0, \quad i = 0, \dots, m + n + 1, \quad j = 1, \dots, n$

$\langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, \dots, m + n + 1,$

where the variables g_i are subgradients of C^{opt}

(for upper bound, in special case, LP is exact)

Robustness

in practice we have **uncertainty**:

- basket weights may change, or not exactly known at present time
- price information may be noisy (e.g., bid-ask)
- strike prices also may vary
- hedging strategies may be implemented with errors

we address the upper bound problem, in the special case

Bid-ask spread

bid-ask spread corresponds to a **box uncertainty** for the vector of observed option prices

$$\underline{p} \leq p \leq \bar{p}$$

the worst-case value of upper (lower) bound is attained at $p^{\text{worst}} = \underline{p}$ (or $p^{\text{worst}} = \bar{p}$)

we may want to introduce "correlation" in the model ... **ellipsoidal uncertainty** on p is as easy

Ellipsoidal uncertainty in basket weights

worst-case upper bound under weight uncertainty is

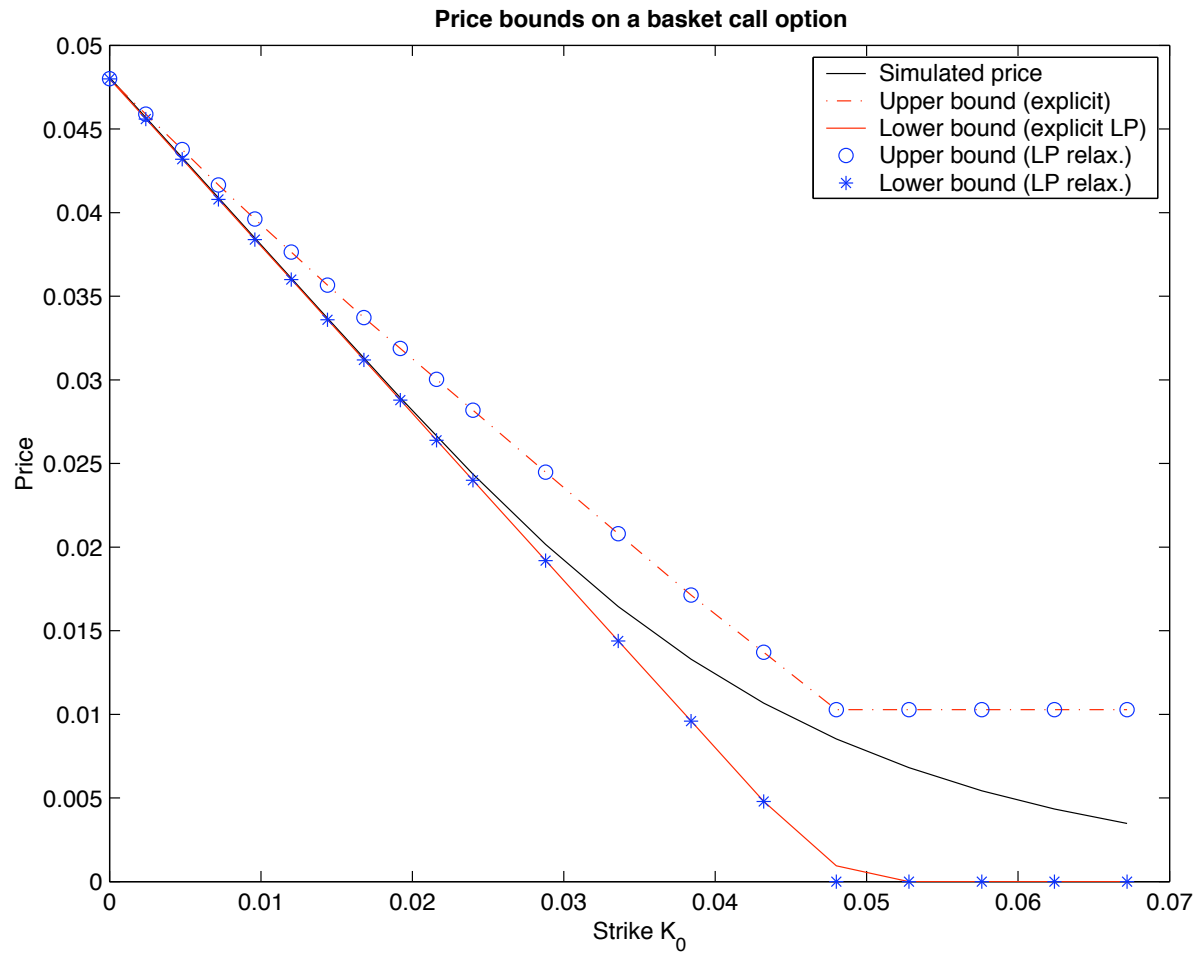
$$p^{\sup} = \max_{w \in \mathcal{E}} w^T p + (w^T K - K_0)_+$$

where $\mathcal{E} = \{\hat{w} + Ru : \|u\|_2 \leq 1\}$ is a given ellipsoid

we have

$$\begin{aligned} p^{\sup} &= \max_{w, t} w^T p + t(w^T K - K_0) : w \in \mathcal{E}, 0 \leq t \leq 1 \\ &= \max_{t=0,1} (tK + p)^T w + \|R^T(tK + p)\|_2 - tK_0 \end{aligned}$$

Example



Robustness for the general case

i have no idea about this

Summary

- general problem important in practice
- special cases yields easy-to-compute bounds
- developed an LP relaxation for general case
- relaxation exact in some special cases

Further research

- imposing smoothness constraints
- multi-period problem (Bertsimas, 2003)
- link with model-based approaches

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for more info: d'Aspremont & El Ghaoui, Static arbitrage bounds for basket option prices, submitted to *Oper. Res.*, 2003