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Static Arbitrage Bounds on Basket Option Prices

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Abstract. We consider the problem of computing upper and lower bounds on the price of an European basket call option, given prices on other similar options. Although this problem is hard to solve exactly in the general case, we show that in some instances the upper and lower bounds can be computed via simple closed-form expressions, or linear programs. We also introduce an efficient linear programming relaxation of the general problem based on an integral transform interpretation of the call price function. We show that this relaxation is tight in some of the special cases examined before.

Key words. Arbitrage, Linear Programming, Radon Transform, Basket Options, Moment Problems.

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Notation

For two n-vectors x, y, $x \geq y$ (resp. $x < y$) means $x_i \geq y_i$ (resp. $x_i < y_i$), $i = 1, \ldots, n$; x_+ denotes the positive part of x, which is the vector with components $\max(x_i, 0)$. e is the n-vector with all components equal to one, and e_i is the i-th unit vector of $\mathbf{R}^n$. The set $\mathbf{R}^n_+$ denotes the set of n-vectors with non-negative components, and $\mathbf{R}^n_{++}$ its interior. The cone of nonnegative measures with support included in $\mathbf{R}^n_+$ is denoted by $\mathcal{K}$. For $w \in \mathbf{R}^m$, $K \in \mathbf{R}$ and $g \in \mathbf{R}^{m+1}$, the notation $\langle g, (w, K) \rangle$ denotes the scalar product $\tilde{g}^T w + g_{m+1} K$, where $\tilde{g}$ contains the first m elements of g.

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1. Introduction

1.1. Problem setup

Let $p \in \mathbf{R}_+^m$, $K_0 \in \mathbf{R}_+$, $w_0 \in \mathbf{R}_{++}^n$ and $K_i \in \mathbf{R}_+$, $w_i \in \mathbf{R}_+^n$, for $i = 1, \ldots, m$. We consider the problem of computing upper (resp. lower) bounds on the price of a European basket call option with maturity T, strike K_0 and weight vector w_0:

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi(w_0^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi(w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m, \end{array} \tag{1}$$

over all probability distributions $\pi \in \mathcal{K}$ on the asset price vector x, consistent with a given set of observed prices p_i of options on other baskets. Note that we implicitly assume that all the options have the same maturity, and that, without loss of generality, the risk-free interest rate is zero (we compare prices in the forward market).

We seek non-parametric bounds, that is, we do not assume any specific model for the underlying asset prices; our sole assumption is the absence of a static or "buy-and-hold" arbitrage today (i.e. an arbitrage that only requires trading today and at the option maturity). The primary objective of these bounds is not to detect and exploit arbitrage opportunities in the basket option market, illiquidity issues are likely to make these opportunities hard to exploit. However, the data on basket prices (index options in equity markets or swaptions in fixed income) is *very sparse* and traders often rely on intuitive guesses to extrapolate the remaining points, using these prices to calibrate models and evaluate more complex derivatives. Our results aim to provide an efficient method to check the validity of these extrapolated prices where they are the most likely to create static arbitrage opportunities, i.e. very far in or out of the money.

From a financial point of view, our approach can be seen as a one-period, non-parametric computation of the upper and lower hedging prices defined in [EKQ91] and [EKQ95] (see also [KS98]). The necessary conditions we detail in section 2 have been extensively used in the unidimensional case to infer information on the state-price density given option prices (see [BL78] or [LL00] among others), we study here a multidimensional generalization.

From an optimization point of view, problems such as the one above have received a significant amount of attention in various forms. First, we can think of problem (1) as a *linear semi-infinite program*, *i.e.* a linear program with a finite number of linear constraints on an infinite dimensional variable. We use this interpretation and the related duality results to compute closed-form solutions in some particular cases. Secondly, we can see (1) as a *generalized moment problem*. This approach was successfully used in dimension one by [BP02], who solve the one dimensional problem completely and show that the multidimensional extension is NP-Hard. However, their relaxation algorithm requires the solution of a number of linear programs that is potentially exponential in n, the number of assets. This makes the method prohibitive for large-scale problems. Finally, as in [HS90], one can think of (1) as an *integral transform inversion problem*. This is the approach we adopt to design an efficient relaxation in the general case.

The contribution of this work is twofold. First, exploiting the necessary convexity of arbitrage free call prices

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)_+,$$

where $C(w, K)$ is the price of a basket call option with weights w and strike K, we detail a relaxation technique providing upper (resp. lower) bounds on the solution to (1). The resulting infinite dimensional linear program on the call price function can be solved exactly. Compared to the decomposition method proposed by [BP02], this relaxation technique has the advantage of being polynomial-time in the number of assets and constraints.

Secondly, in some particular cases, we provide exact solutions to (1) that have a polynomial complexity in the number of assets and constraints. We also obtain expressions for the corresponding pricing measures, and use them later on to prove tightness of the linear programming relaxation in the general case. [LW03] and [LW04] derived independently an equivalent upper bound in the same particular cases and exact lower bounds in dimension 2.

Finally, using recent results by [HLW04] who compute an optimal solution to (1) in the particular case where one seeks an upper bound on the price of a basket option with positive weights given only single asset option prices:

$$\begin{array}{ll} \text{maximize} & \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi (x_i - K_i^j)_+ = p_i^j, \quad i = 1, \ldots, m, \;\; j = 1, \ldots, J^i, \end{array} \tag{2}$$

we show that our linear programming relaxation is tight.

Our paper is organized as follows. We begin in section 1.2 by a brief reminder of the fundamental duality between pricing measures and positive portfolios. In section 2 we detail a relaxation for problem (1) using shape constraints on the call price $C(w, K)$ as a function of the weight vector w and the strike price K. Using linear programming duality arguments, we obtain in section 3 closed-form formulas (or simple linear programs) giving upper and lower bounds in some particular cases. In section 4, using [HLW04] we show that the linear programming relaxation derived in section 2 is tight in the particular case (2) above. Finally, section 5 provides some numerical examples.

1.2. Semi-infinite programming duality

We begin by detailing a key duality result linking the existence of a pricing measure (or state price density in [Duf96]) and the absence of an arbitrage portfolio. In the general case, we can write the upper bound problem as a semi-infinite program:

$$p^{\sup} := \sup_{\pi \in \mathcal{K}} \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx \text{ subject to } \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx = p, \;\; \int_{\mathbf{R}_+^n} \pi(x)dx = 1, \tag{3}$$

where

$$\psi(x) := (w_0^T x - K_0)_+, \;\; \phi_i(x) := (w_i^T x - K_i)_+, \;\; i = 1, \ldots, m.$$

We define the Lagrangian (on $\mathcal{K} \times \mathbf{R}^{m+1}$):

$$L(\pi, \lambda, \lambda_0) = \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx + \lambda^T \left(p - \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx \right) + \lambda_0 (1 - \int_{\mathbf{R}_+^n} \pi(x)dx),$$

and, as in [HK93], we can write the dual of (3) as:

$$\begin{aligned} d^{\sup} &:= \inf_{\lambda_0, \lambda} : \;\; \lambda^T p + \lambda_0 \;\; : \;\; \lambda^T \phi(x) + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \\ &= \inf_{\lambda} : \;\; \sup_{x \geq 0} : \lambda^T p + \psi(x) - \lambda^T \phi(x). \end{aligned} \tag{4}$$

Both primal and dual problems have very intuitive financial interpretations. The primal problem looks for a pricing measure that maximizes the target option price while satisfying the pricing constraints imposed by the current market conditions. The dual problem looks for the least expensive portfolio of options and cash, $\lambda^T \phi(x) + \lambda_0$, that dominates the option payoff $\psi(x)$. Of course, the dual problem above yields an upper bound on the upper bound.

Similarly, the computation of the lower bound involves

$$p^{\inf} := \inf_{\pi \in \mathcal{K}} \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx \text{ subject to } \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx = p, \;\; \int_{\mathbf{R}_+^n} \pi(x)dx = 1, \tag{5}$$

whose dual is

$$\begin{aligned} d^{\inf} &:= \sup_{\lambda_0, \lambda} : \lambda^T p + \lambda_0 \;\; : \;\; \lambda^T \phi(x) + \lambda_0 \leq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \\ &= \sup_{\lambda} : \inf_{x \geq 0} : \lambda^T p + \psi(x) - \lambda^T \phi(x). \end{aligned} \tag{6}$$

Here, the dual problem provides a lower bound on the lower bound.

General results on semi-infinite linear programs establish the equivalence between the primal and dual formulations. We cite here a sufficient constraint qualification condition for perfect duality from [HK93], which makes an assumption about the support of optimal distributions. (We focus now on the lower bound; a similar result holds for the upper bound problem.)

Proposition 1. *Assume that in problem (6), the support of the asset price distribution can be restricted to a given compact set* $B \subset \mathbf{R}_+^n$. *Assume further that there exists a pair* $(\lambda_0, \lambda) \in \mathbf{R}^{n+1}$ *such that:*

$$\lambda^T \phi(x) + \lambda_0 < \psi(x) \textit{ for all } x \in B.$$

Then if $d^{\inf}$ *is finite, perfect duality holds, namely* $d^{\inf} = p^{\inf}$.

Proof. See [HK93]. □

This constraint qualification condition trivially holds when $\phi(x)$ and $\psi(x)$ are Call option payoffs hence we have $d^{\inf} = p^{\inf}$, provided that the support of distributions feasible for our problem can be restricted to some compact $B \subset \mathbf{R}_+^n$. However, this is often not the case for the bounds detailed in section 3 and we will prove perfect duality directly whenever possible.

2. Relaxation for the general case using an integral transform

2.1. The Radon transform

Let us come back to problem (1), for $p \in \mathbf{R}_+^m$, $K \in \mathbf{R}_+^m$, $w_0 \in \mathbf{R}_{++}^n$, $w_i \in \mathbf{R}_+^n$, $i = 1, ..., m$ and $K_0 \geq 0$, we seek to compute upper and lower bounds on the price of a European call basket option with strike K_0 and weight vector w_0:

$$\mathbf{E}_\pi (w_0^T x - K_0)_+,$$

with respect to all probability distributions $\pi \in \mathcal{K}$ on the asset price vector x, consistent with a given set of m observed prices p_i of options on other baskets and forward prices $q_i = \mathbf{E}_\pi x_j$, that is, given

$$\mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m \quad \text{and} \quad \mathbf{E}_\pi x_j = q_j, \quad j = 1, \ldots, n.$$

Extending to basket options the results of [BL78], we write, for some $\pi \in \mathcal{K}$:

$$\begin{aligned} C(w, K) &= \mathbf{E}_\pi (w^T x - K)_+ \\ &= \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x), \end{aligned}$$

we can think of $C(w, K)$ as a particular integral transform of the measure π and we can try to compute its inverse. If we assume that the measure π is absolutely continuous with respect to the Lebesgue measure with density $\pi(x)$, then for almost all K we have:

$$\frac{\partial^2 C(w, K)}{\partial K^2} = \int_{\mathbf{R}_+^n} \delta(w^T x - K) \pi(x) dx,$$

where $\delta(x)$ is the Dirac Delta function. This means that $\partial^2 C(w, K)/\partial K^2$ is the Radon transform of the measure π (see [Hel99] for example). In this setting, the general pricing problem above can then be rewritten as the following infinite dimensional problem:

$$\begin{array}{ll} \text{min./max.} & f(w_0, K_0) \\ \text{subject to} & f(w_j, K_j) = p_j, \quad j = 1, \ldots, m \\ & f(e_i, 0) = q_i, \quad i = 1, \ldots, n \\ & f(w, K) \in \mathcal{R}_C, \end{array}$$

where e_i is the Euclidean basis in $\mathbf{R}^n$ and $\mathcal{R}_C$ is the range of the (linear) integral transform

$$\begin{aligned} C : \mathcal{K} &\to \mathcal{R}_C \\ \pi &\mapsto C(w,K) = \textstyle\int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x). \end{aligned}$$

Thus, the problem of finding all possible arbitrage-free option prices becomes equivalent to that of characterizing the range of the Radon transform on the set of nonnegative measures $\mathcal{K}$. This has been done by [HS90] in the context of production functions (which can be thought of as Put options). As in [HS90], we denote by $C^\infty\{0, \mathbf{R}_+^n\}$ the set of functions f such that for any k there is a polynomial $P_k(x)$ of degree k such that:

$$f(x) - P_k(x) = o(|x|^k), \quad \text{as } |x| \to 0, \ x \in \mathbf{R}_+^n.$$

Using Call-Put parity, we can directly derive from [HS90, theorem 3.2] the following result:

Proposition 2. *A function $C(w,K)$, with $w \in \mathbf{R}_+^n$ and $K > 0$ belongs to $\mathcal{R}_C$, i.e. it can be represented in the form*

$$C(w,K) = \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x),$$

where π is a nonnegative measure on a compact of $\mathbf{R}_+^n$, if and only if the following conditions hold.

- *$C(w,K)$ is convex and homogenous of degree one;*
- *for every $w \in \mathbf{R}_{++}^n$, we have*

$$\lim_{K\to\infty} C(w,K) = 0 \ \text{and} \ \lim_{K\to 0^+} \frac{\partial C(w,K)}{\partial K} = -1;$$

- *if we write $D_\xi = \sum_i \xi_i \partial/\partial x_i$, the function*

$$F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w,K)}{\partial K}\right)$$

  belongs to $C^\infty\{0, \mathbf{R}_+^n\}$ and for some $\tilde{w} \in \mathbf{R}_+^n$ the inequalities:

$$(-1)^{k+1} D_{\xi_1} ... D_{\xi_k} F(\lambda \tilde{w}) \geq 0$$

  hold for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all $\xi_1, ..., \xi_k$ in $\mathbf{R}_+^n$.

Proof. See [HS90]. □

This result generalizes the necessary conditions for the absence of arbitrage used by [BL78], [LL00] or [BP02] in dimension one.

2.2. Linear programming relaxation

The conditions above are not tractable in the general case but we can formulate a relaxation of the original program by replacing the last (moment) condition with weaker monotonicity and linearity conditions. We then get an upper bound on the upper bound (resp. a lower bound on the lower bound) solution by computing:

$$\begin{array}{ll} \sup / \inf & C(w_0, K_0) \\ \text{subject to} & C(w,K) \text{ (jointly) convex in } (w,K) \\ & C(w,K) \text{ homogeneous of degree 1} \\ & -1 \leq \partial C(w,K)/\partial K \leq 0 \text{ and } C(w,K) \text{ nondecreasing in } w \\ & C(w_i, 0) = w_i^T q,\ i = 1, ..., m \\ & C(w_i, K_i) = p_i,\ i = 1, ..., m. \end{array} \tag{7}$$

where the variable is here $C(w,K) \in C\left(\mathbf{R}^{n+1} \to \mathbf{R}_+\right)$. As we show below, this infinite program can be reduced to a finite LP. If we define $p_{m+i} = w_i^T q$ and $K_{m+i} = 0$ for $i = 1, ..., m$ and $p_{2m+1} = w_0^T q$ with $K_{2m+1} = 0$, we can show the following result:

Proposition 3. *If the following finite LP:*

$$\begin{array}{ll} \textit{max./min.} & p_0 \\ \textit{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i, \quad i, j = 0, ..., 2m+1 \\ & g_{i,j} \geq 0, \ -1 \leq g_{i,n+1} \leq 0, \quad i = 0, ..., 2m+1, \quad j = 1, ..., n \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, ..., 2m+1, \end{array} \tag{8}$$

in the variables $p_0 \in \mathbf{R}_+$ and $g_i \in \mathbf{R}^{n+1}$ for $i = 0, ..., 2m+1$, is strictly feasible and its optimal value is finite, the infinite program (7) and its discretization (8) have the same optimal value. Furthermore, an optimal point of (7) can be constructed from the solution to (8).

Proof. As in [BV04], we first notice that as a discretization of the infinite program (7), the finite LP will compute a lower (or upper) bound on its optimal value. Let us now show that this bound is in fact equal to the optimal value of (7). If we note $z^* = \left[p_0^*, g_0^{*T}, \ldots, g_k^{*T}\right]^T$ the optimal solution to the LP problem above and if we define:

$$C(w,K) = \max_{i=0,\ldots,2m+1} \left\{ p_i^* + \langle g_i^*, (w,K) - (w_i, K_i) \rangle \right\},$$

where $p_i^* = p_i$ for $i = 1, \ldots, 2m+1$. $C(w,K)$ satisfies

$$C(w_i, K_i) = p_i, \quad i = 1, \ldots, 2m+1,$$

and, by construction, $C(w_0, K_0)$ attains the lower bound p_0 computed in the finite LP. Also, $C(w,K)$ is convex as the pointwise maximum of affine functions and is piecewise affine with gradient g_i, which implies that it also satisfies the convexity and monotonicity conditions in (7), hence it is a feasible point of the infinite dimensional problem. This means that both problems have the same optimal value and $C(w,K)$ is an optimal solution to the infinite dimensional program in (7). □

3. Exact upper and lower bounds using LP duality

In this section, we address the problem of computing exact bounds in some particular cases using linear programming duality. We first consider the case when the observed data set corresponds to option and forward prices on each individual asset since in practice, observations always include the forward contract prices $\mathbf{E}_\pi x_i = q_i$, $i = 1, \ldots, n$ (forward contracts exist whenever options do). The expressions derived in these simple particular cases will be used in the section 4 to show tightness of the upper bound relaxation in the general case.

We first examine the problem of computing upper and lower bounds on

$$\mathbf{E}_\pi (w^T x - K_0)_+,$$

given the $2n$ constraints

$$\mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad \mathbf{E}_\pi x_i = q_i, \quad i = 1, \ldots, n, \tag{9}$$

where $K_0 > 0$ and w, K, p, q are given vectors of $\mathbf{R}^n_{++}$. We will assume that $0 \leq p < q \leq p + K$, which is a necessary and sufficient condition for the problem above to be feasible. We show sufficiency by constructing a discrete asset price distribution that matches these prices. Let us define marginal distributions $\pi_i(x_i)$ such that

$$x_i = \begin{cases} 0 & \text{with probability } \pi_i(0) = 1 - \frac{q_i - p_i}{K_i} \\ \frac{q_i K_i}{q_i - p_i} & \text{with probability } \pi_i(\frac{q_i K_i}{q_i - p_i}) = \frac{q_i - p_i}{K_i} \end{cases} \tag{10}$$

and because $0 \leq p < q \leq p + K$, we know that

$$0 < \frac{q_i - p_i}{K_i} \leq 1.$$

Using Sklar's theorem, we can then construct an asset distribution $\pi(x)$ with marginals $\pi_i(x_i)$, *i.e.* matching the market prices of single asset options, hence the market is arbitrage free. From the form of the constraints in (9), we also observe that the constraints $0 \leq p < q \leq p + K$ are necessary.

3.1. Upper bound

3.1.1. One forward and one option price constraint per asset Here, we apply the semi-infinite duality result to the upper bound problem described in (9) to show the following result:

Proposition 4. *Let* $p,\ q \in \mathbf{R}^n_+$, $K_0 \in \mathbf{R}_+$, $w \in \mathbf{R}^n_{++}$ *and* $K_i \in \mathbf{R}_+$ *for* $i = 1, \ldots, n$, *with* $0 \leq p_i < q_i \leq p_i + K_i$. *An upper bound on the optimal value of the problem:*

$$\begin{array}{ll} \textit{maximize} & \mathbf{E}_\pi (w^T x - K_0)_+ \\ \textit{subject to} & \mathbf{E}_\pi (x_i) = q_i \\ & \mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad i = 1, \ldots, n, \end{array}$$

is given by:

$$d^{\sup} = \max_{0 \le j \le n+1} w^T p + \sum_i w_i \min(q_i - p_i, \beta_j K_i) - \beta_j K_0, \tag{11}$$

with $\beta_j := (q_j - p_j)/K_j \in [0,1]$, $j = 1, \dots, n$ *and the convention* $\beta_0 = 0$, $\beta_{n+1} = 1$.

Proof. In view of the general result (4), the dual problem can be expressed as

$$d^{\sup} = \inf_{\lambda+\mu \ge w} \sup_{x \ge 0} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x - K)_+ - \mu^T x, \tag{12}$$

where, without loss of generality, we have included the constraint $\lambda + \mu \ge w$, in order to ensure that the inner supremum is finite. We introduce a partition of $\mathbf{R}^n_+$ as follows. To a given subset I of $\{1, \dots, n\}$, we associate a subset D_I of $\mathbf{R}^n_+$, defined by

$$D_I = \{x \ : \ x_i > K_i, \ i \in I, \ \ 0 \le x_i \le K_i, \ i \in I^c\},$$

where I^c denotes the complement of I in $\{1, \dots, n\}$. For $z \in \mathbf{R}^n$, let z_I be the vector formed with the elements $(z_i)_{i \in I}$, in the ascending order of indices in I.

We have for

$$\begin{aligned} d^{\sup} = & \inf_{\lambda+\mu \ge w} : \max_{t \in \{0,1\}, I \subseteq \{1,\dots,n\}} : \sup_{x \in D_I} : \lambda^T p + \mu^T q \\ & + t(w^T x - K_0) - \lambda_I^T (x_I - K_I) - \mu^T x \\ = & \inf_{\lambda+\mu \ge w} : \max_{t \in \{0,1\} I \subseteq \{1,\dots,n\}} : \lambda^T p + \mu^T q + h(\lambda, \mu, I, t), \end{aligned}$$

where $h(\lambda, \mu, I, t)$ is given by

$$\begin{aligned} h(\lambda, \mu, I, t) := & \sup_{x \in D_I} : t(w^T x - K_0) - \lambda_I^T (x_I - K_I) - \mu^T x \\ = & \sup_{0 \le x_{I^c} \le K_{I^c}} : (t w_{I^c} - \mu_{I^c})^T x_{I^c} - tK_0 + \lambda_I^T K_I \\ & + \sup_{x_I > K_I} : (t w_I - \mu_I - \lambda_I)^T x_I \\ = & \begin{cases} (t w_{I^c} - \mu_{I^c})_+^T K_{I^c} - tK_0 + (t w_I - \mu_I)^T K_I & \text{if } \lambda_I + \mu_I \ge t w_I, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

We note that finiteness of $h(\lambda, \mu, I, t)$ is guaranteed by $\lambda + \mu \ge w$ and $0 \le t \le 1$. When these conditions hold, the maximum value of $h(\lambda, \mu, I, t)$ over $I \subseteq \{1, \dots, n\}$ is obtained when the complement I^c is the full set, that is, when I is empty. We obtain

$$\max_{I \subseteq \{1,\dots,n\}} h(\lambda, \mu, I, t) = (tw - \mu)_+^T K - tK_0.$$

Optimizing over t, we obtain

$$\max_{t \in \{0,1\}} \max_{I \subseteq \{1,\dots,n\}} h(\lambda, \mu, I, t) = \max\left((-\mu)_+^T K, (w - \mu)_+^T K - K_0\right).$$

This results in the following expression for $d^{\sup}$:

$$\begin{aligned} d^{\sup} &= \inf_{\lambda+\mu\geq w} \lambda^T p + \mu^T q + \max\left((-\mu)_+^T K, (w-\mu)_+^T K - K_0\right) \\ &= \inf_{\mu} w^T p + \mu^T (q-p) + \max\left((-\mu)_+^T K, (w-\mu)_+^T K - K_0\right), \end{aligned} \tag{13}$$

which admits the following linear programming representation:

$$d^{\sup} = \inf_{\mu,t,v,z} : w^T p + \mu^T (q-p) + t \quad \begin{array}{l} t \geq v^T K, \;\; v \geq 0, \;\; v+\mu \geq 0 \\ t \geq z^T K - K_0, \;\; z \geq 0, \;\; z+\mu \geq w. \end{array}$$

The problem is feasible, and is thus equivalent to its dual. After some elimination of dual variables, the dual problem can be rewritten as:

$$d^{\sup} = \max_{y,\beta} w^T p + w^T y - \beta K_0 \;\;:\;\; \begin{array}{l} (1-\beta)K \geq q-p-y \geq 0 \\ \beta K \geq y \geq 0. \end{array}$$

We remark that the above problem is feasible if and only if $p < q \leq p + K$. We thus recover the primal feasibility condition mentioned before. This condition ensures that the dual bound $d^{\sup}$ is finite. The above further reduces to the one-dimensional problem:

$$d^{\sup} = \max_{0\leq\beta\leq 1} : w^T p + \sum_i w_i \min(q_i - p_i, \beta K_i) - \beta K_0. \tag{14}$$

The above problem is the maximization of a piecewise linear concave function of one variable, thus the maximum is attained at one of the break points $\beta_j := (q_j - p_j)/K_j \in [0,1]$, $j = 1, \ldots, n$, or for $\beta = 0, 1$. This way, we can obtain a closed-form expression for the upper bound, namely

$$d^{\sup} = \max_{0\leq j\leq n+1} w^T p + \sum_i w_i \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

with the convention $\beta_0 = 0$, $\beta_{n+1} = 1$, which is the desired result. □

Remark that, when the price of forwards is not given , the upper bound is readily obtained by setting the variable μ, which is the variable dual to the constraint $\mathbf{E}_\pi x = q$, to zero in the expression (13). We get the simple closed-form expression

$$d^{\sup} = w^T p + (w^T K - K_0)_+, \tag{15}$$

which can be obtained as a direct consequence of Jensen's inequality applied to the function $x \to x_+$. We can check that the above bound satisfies some basic properties: it is convex in w and concave in p, q. Also, when $w = e_i$ (the i-th unit vector), and $K_0 = K_i$, we obtain $d^{\sup} = p_i$, while for $K_i = 0$ (*i.e.* the options are in fact forward contracts), we obtain $d^{\sup} = p_i (= q_i)$.

3.1.2. Two option price constraints per asset Using [HLW04], the result we just obtained can directly be extended to the (slightly) more general case where two option price constraints are given for each asset (but no forward price is specified). We will use this technical result to show tightness of the LP relaxation in section 4.

We let $p^1, p^2 \in \mathbf{R}^n_+$, $K_0 \in \mathbf{R}_+$, $w \in \mathbf{R}^n_{++}$ and $K^1, K^2 \in \mathbf{R}^n_+$ (with $K^2 > K^1$). We look for an upper bound on the optimal value of the problem:

$$\begin{array}{ll} \text{maximize} & \mathbf{E}_\pi(w^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi(x_i) = q_i \\ & \mathbf{E}_\pi(x_i - K_i^j)_+ = p_i^j, \quad i = 1, \ldots, n, \; j = 1, 2. \end{array} \tag{16}$$

where $q \in \mathbf{R}^n_+$ is here a *variable*. From (7), to preclude arbitrage between options and forwards, we must impose:

$$\underline{q}_i := p_i^1 + K_i^1 \frac{p_i^1 - p_i^2}{K_i^2 - K_i^1} \leq q_i \leq p_i^1 + K_i^1 := \overline{q}_i, \quad i = 1, \ldots, n.$$

As in [HLW04], for each asset x_i, we can then form $\bar{C}^{(i)}(K_i^j)$, the largest decreasing convex function such that $\bar{C}^{(i)}(0) = p_i^1 + K_i^1$ and $\bar{C}^{(i)}(K_i^j) = p_i^j$ for $i = 1, \ldots, n,\; j = 1, 2$. Then every maximal decreasing convex function $C^{(i)}(k)$ matching the market prices can be written in terms of q_i as:

$$C^{(i)}(k) = \bar{C}^{(i)}(k) - (p_i^1 + K_i^1 - q_i)\frac{(K_i^1 - k)_+}{K_i^1}, \quad i = 1, \ldots, n.$$

From [HLW04], we know that for a given q the upper bound on (16) can be written:

$$\inf_{\{\lambda \geq 0,\; \lambda^T e = 1\}} \sum_{i=1}^n w_i C^{(i)}(\frac{\lambda_i}{w_i} K_0)$$

this means that an upper bound on the solution to (16) for all possible values of q can be found by solving:

$$\sup_{\{\underline{q}_i \leq q_i \leq \overline{q}_i\}} \inf_{\{\lambda \geq 0,\; \lambda^T e = 1\}} \sum_{i=1}^n w_i \bar{C}^{(i)}\left(\frac{\lambda_i}{w_i} K_0\right) - (p_i^1 + K_i^1 - q_i)\frac{(K_i^1 - (\frac{\lambda_i}{w_i} K_0))_+}{K_i^1}$$

and because both $\bar{C}^{(i)}$ and $(K_i^1 - (\frac{\lambda_i}{w_i} K_0))_+$ are convex functions of λ (with $(p_i^1 + K_i^1 - q_i) \leq 0$), this is also:

$$\inf_{\{\lambda \geq 0,\; \lambda^T e = 1\}} \sup_{\{\underline{q}_i \leq q_i \leq \overline{q}_i\}} \sum_{i=1}^n w_i \bar{C}^{(i)}\left(\frac{\lambda_i}{w_i} K_0\right) - (p_i^1 + K_i^1 - q_i)\frac{(K_i^1 - (\frac{\lambda_i}{w_i} K_0))_+}{K_i^1}.$$

The inner supremum is reached for $q_i = p_i^1 + K_i$ and we get the solution to (16) as:

$$\inf_{\{\lambda \geq 0,\; \lambda^T e = 1\}} \sum_{i=1}^n w_i \bar{C}^{(i)}\left(\frac{\lambda_i}{w_i} K_0\right).$$

If $q_i = p_i^1 + K_i$, the corresponding measure in [HLW04] places no weight on values of x_i smaller than K_i^1, hence the problem with two options:

$$\begin{array}{ll}\text{maximize} & \mathbf{E}_\pi(w^T x - K_0)_+\\ \text{subject to} & \mathbf{E}_\pi(x_i - K_i^j)_+ = p_i^j, \quad i = 1, \ldots, n, \; j = 1, 2.\end{array}$$

is equivalent to the following problem (setting $x = K^1 + y$):

$$\begin{array}{ll}\text{maximize} & \mathbf{E}_\pi(w^T y - (K_0 - w^T K^1))_+\\ \text{subject to} & \mathbf{E}_\pi(y_i) = p_i^1\\ & \mathbf{E}_\pi(y_i - (K_i^2 - K_i^1))_+ = p_i^2, \quad i = 1, \ldots, n.\end{array}$$

where one forward and one option price constraint are given per asset.

3.2. Perfect duality: upper bound

We first compute the optimal probability measures corresponding to the upper bound result with option and forward price constraints obtained in section 3.1.1. We can recover an optimal distribution, or a sequence of distributions which achieve the bound in the limit. This provides a direct proof of the fact that $p^{\text{sup}} = d^{\text{sup}}$, i.e. that the upper bound computed in Proposition 4 is tight.

Without loss of generality, we assume $e^T w = 1$. In (14) we obtained:

$$d^{\text{sup}} = \sup_{0 \leq \beta \leq 1} : w^T p + \sum_i w_i \min(q_i - p_i, \beta K_i) - \beta K_0,$$

which can be rewritten (the min is taken elementwise):

$$\sup_{0 \leq \beta \leq 1} \; w^T \min\{q - \beta K_0 e, p + \beta\,(K - K_0)\},$$

or again:

$$\sup_{0 \leq \beta \leq 1} \inf_{t \in [0,1]^m} \; w^T((e - t)(q - \beta K_0 e) + t(p + \beta\,(K_i - K_0))).$$

Using LP duality we know that this is also equal to (with $e^T w = 1$):

$$\inf_{t \in [0,1]^m} \sup_{0 \leq \beta \leq 1} \; \beta\left(w^T t K - K_0\right) + w^T(e - t)q + w^T t p.$$

We express the above as

$$\inf_{t \in [0,1]^m} \; w^T(e - t)q + w^T t p + \left(w^T t K - K_0\right)_+.$$

This problem can be solved exactly as a finite linear program, and we obtain t^* such that:

$$d^{\text{sup}} = w^T\left((e - t^*)q + t^* p\right) + \left(w^T t^* K - K_0\right)_+. \tag{17}$$

We recognize here the expression of the upper bound on the price of a basket, where we are only given the following option price constraints:

$$\mathbf{E}_\pi(x_i - \hat{K}_i)_+ = \hat{p}_i, \;\; i = 1, \ldots, n,$$

where $\hat{K} := t^* K$ and $\hat{p} := (e - t^*)q + t^* p$. In other words, as in [HLW04], this upper bound is equal to the bound we would obtain given only one (synthetic) call price $\hat{p}_i$ per asset corresponding to a strike price $\hat{K}_i$.

Suppose first that $(w^T K \leq K_0)$, in this case we have $t^* = e$ and $d^{\sup} = w^T p$, and as in [HLW04, p.15, Case 3] we can construct a sequence of distributions converging to the optimal value. If however $(w^T K > K_0)$ and $p > 0$, then using [HLW04, p.14, Case 2], we know that the bound is attained by a distribution with finite support. Finally, if $(w^T K > K_0)$ and $p_i = 0$ for some index i, [HLW04, p.12, Case 1] shows that a similar result holds.

Finally, using the result above, we know that this result is still valid when we replace the option and forward price constraints with two option price constraints.

3.3. Lower bound

We can obtain a similar result for the lower bound problem. In this case however the solution is not in closed form and involves solving a (polynomial size) linear program.

Proposition 5. *Let* $p,\; q \in \mathbf{R}^n_+$*,* $K_0 \in \mathbf{R}_+$*,* $w \in \mathbf{R}^n_{++}$ *and* $K_i \in \mathbf{R}_+$ *for* $i = 1, \ldots, n$*, with* $0 \leq p_i < q_i \leq p_i + K_i$*. A lower bound on the optimal value of the problem:*

$$\begin{array}{ll} \textit{minimize} & \mathbf{E}_\pi(w^T x - K_0)_+ \\ \textit{subject to} & \mathbf{E}_\pi(x_i) = q_i \\ & \mathbf{E}_\pi(x_i - K_i)_+ = p_i, \quad i = 1, \ldots, n, \end{array}$$

can be computed by solving the following linear program:

$$\begin{array}{rll} d^{\inf} = & \sup\limits_{\lambda,\mu,\alpha_0,\ldots,\alpha_n} & \lambda^T p + \mu^T (q - K) + h \\ & \textit{subject to} & \lambda + \mu \leq w \\ & & h \leq \alpha_0 (w^T K - K_0) - (\alpha_0 w - \mu)_+^T K, \; 0 \leq \alpha_0 \leq 1 \\ & & \forall i \;:\; h \leq \alpha_i (w^T K - K_0) - \sum_{j \neq i} (\alpha_i w_j - \mu_j)_+ K_j \\ & & (\lambda_i + \mu_i)_+ / w_i \leq \alpha_i \leq 1, \end{array} \tag{18}$$

which has $3n + 2$ *variables and* $3n + 4$ *constraints.*

Proof. In the lower bound case, the dual problem is

$$d^{\inf} = \sup_{\lambda + \mu \leq w} \inf_{x \geq 0} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x - K)_+ - \mu^T x,$$

where we exploited the fact that the inner infimum is $-\infty$ unless $\lambda + \mu \leq w$.

Let us use the same notation as before. We have

$$\begin{aligned} d^{\inf} = & \sup_{\lambda+\mu\leq w} : \min_{I\subseteq\{1,\ldots,n\}} : \inf_{x\in D_I} : \lambda^T p + \mu^T q \\ & +(w^T x - K_0)_+ - \lambda_I^T(x_I - K_I) - \mu^T x \\ = & \sup_{\lambda+\mu\leq w} : \min_{I\subseteq\{1,\ldots,n\}} : \lambda^T p + \mu^T q + h(\lambda,\mu,I), \end{aligned}$$

where

$$h(\lambda,\mu,I) = \inf_{x,y_0} \; y_0 - \lambda_I^T(x_I - K_I) - \mu^T x \;:\; x \in D_I,\; y_0 \geq w^T x - K_0,\; y_0 \geq 0.$$

We have by linear programming duality

$$h(\lambda,\mu,I) = \sup\,(\alpha w - \mu)^T K - \alpha K_0 - (\alpha w_{I^c} - \mu_{I^c})_+^T K_{I^c} \;:\; \begin{array}{l} \alpha w_I - \lambda_I - \mu_I \geq 0 \\ 0 \leq \alpha \leq 1 \end{array}$$

Thus

$$d^{\inf} = \sup_{\lambda+\mu\leq w} \lambda^T p + \mu^T(q-K) + \min_{I\subseteq\{1,\ldots,n\}} f(\lambda,\mu,I),$$

where

$$f(\lambda,\mu,I) := \sup_{\underline{\alpha}(\lambda,\mu,I)\leq\alpha\leq 1} \alpha(w^T K - K_0) - (\alpha w_{I^c} - \mu_{I^c})_+^T K_{I^c},$$

and

$$\underline{\alpha}(\lambda,\mu,I) := \max_{i\in I} \frac{(\lambda_i+\mu_i)_+}{w_i},$$

with the convention that $\underline{\alpha}(\lambda,\mu,I) = 0$ when I is empty.

Let I be a non-empty subset of $\{1,\ldots,n\}$. Let $i \in \arg\max_{i\in I}(\lambda_i+\mu_i)_+/w_i$. We observe that

$$\underline{\alpha}(\lambda,\mu,I) = \underline{\alpha}(\lambda,\mu,\{i\}),$$

and

$$f(\lambda,\mu,I) \geq f(\lambda,\mu,\{i\}),$$

which dramatically reduces the complexity of the minimization subproblem: instead of computing the minimum over all 2^n sets $I \subseteq \{1,\ldots,n\}$ it is sufficient to pick I in the set of *singletons* of $\{1,\ldots,n\}$, or $I = \emptyset$. Therefore, the problem reads as a linear program

$$\begin{array}{rll} d^{\inf} = & \sup\limits_{\lambda,\mu,\alpha_0,\ldots,\alpha_n} & \lambda^T p + \mu^T(q-K) + h \\ & \text{subject to} & \lambda + \mu \leq w \\ & & h \leq \alpha_0(w^T K - K_0) - (\alpha_0 w - \mu)_+^T K,\; 0 \leq \alpha_0 \leq 1 \\ & & \forall i \;:\; h \leq \alpha_i(w^T K - K_0) - \sum_{j\neq i}(\alpha_i w_j - \mu_j)_+ K_j \\ & & (\lambda_i+\mu_i)_+/w_i \leq \alpha_i \leq 1, \end{array} \tag{19}$$

and can be solved efficiently, since it has $O(n)$ constraints and variables. □

3.4. Perfect duality: lower bound without forwards

Here we study a particular case of the lower bound problem discussed in section 3.3 where we are not given information on forward prices. We can't prove perfect duality in the setting of section 3.3 but prove it below in a more restrictive case. Without information on the forward prices, we simply set the dual variable μ to zero in expression (19) to obtain:

$$d^{\inf} = \sup_{0 \le \xi \le e} \; w^T p\xi + h \;:\; h \le 0,\; h \le \xi_i(w_i K_i - K_0),\; 1 \le i \le n. \tag{20}$$

We note that $d^{\inf}$ can now be expressed as the solution of a non-linear, convex optimization problem:

$$d^{\inf} = \sup_{\xi} \; w^T p\xi - \max_{1 \le i \le n} \; \xi_i (K_0 - w_i K_i)_+ \;:\; 0 \le \xi \le e, \tag{21}$$

or from its dual:

$$d^{\inf} = \inf_{\nu} \sum_{i=1}^{n} \left(p_i w_i - \nu_i (K_0 - w_i K_i)_+\right)_+ \;:\; \nu^T e = 1,\; \nu \ge 0. \tag{22}$$

We can reduce again this optimization problem to a line search over a scalar parameter, by elimination of the variable ξ. We obtain

$$d^{\inf} = \sum_{\{i \,:\, K_i w_i \ge K_0\}} p_i w_i + \sup_{v \ge 0} \sum_{\{i \,:\, K_i w_i < K_0\}} p_i w_i \min(1, \frac{v}{K_0 - K_i w_i}) - v.$$

The minimization above can be further reduced to a closed-form expression by noting that the piecewise-linear function (of v) involved has break points at $\gamma_i = K_0 - K_i w_i$ (for i such that $\gamma_i > 0$) and 0. We obtain as before a lower bound on the minimization problem

$$\begin{array}{ll} \text{minimize} & \mathbf{E}_\pi (w^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad i = 1, \ldots, n, \end{array}$$

as:

$$\begin{aligned} d^{\inf} = & \sum_{\{i \,:\, K_i w_i \ge K_0\}} p_i w_i \\ & + \max_{\{j \,:\, K_j w_j < K_0\}} \left(\sum_{\{i \,:\, K_i w_i < K_0\}} p_i w_i \min(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}) - K_0 + w_j K_j \right)_+ \end{aligned}$$

Now, the linear programming expression (22) allows us to recover an optimal asset price distribution or a sequence of distributions that are optimal in the limit, as follows. Let ν be an optimal vector for problem (22). Let $\mathcal{I}$ be the set of indices i such that $K_0 > w_i K_i$. We note that $i \notin \mathcal{I}$ implies $\nu_i = 0$. For simplicity we assume that $\mathcal{I} = \{1, \ldots, m\}$, where $0 \le m \le n$ (the choice $m = 0$ corresponding to empty $\mathcal{I}$).

First we examine the case when $m = 0$, that is, $\mathcal{I}$ is empty. In other words, $\min_i w(i)K_i \geq K_0$, and therefore $d^{\inf} = p^T w$. For $\epsilon > 0$, we can define an asset price distribution π_ϵ such that

$$x = \begin{cases} \epsilon^{-1}p + K & \text{with probability } \pi_\epsilon = \epsilon \\ 0 & \text{with probability } \pi_= 1 - \epsilon \end{cases} \tag{23}$$

which satisfies the pricing constraints and:

$$\mathbf{E}_{\pi_\epsilon}[(w_0^T x - K_0)_+] = w^T p + \epsilon(K - K_0),$$

we recover the lower bound by taking the limit when ϵ goes to zero.

Next, we assume $m \geq 1$. Let $\alpha = (n - m)/m$. For ϵ such that

$$\epsilon < \alpha^{-1} \min_{1 \leq i \leq m} \nu_i(\neq 0),$$

we define the vector $\nu(\epsilon)$ by

$$\nu_i(\epsilon) = \begin{cases} \nu_i - \alpha\epsilon & \text{if } 1 \leq i \leq m, \\ \epsilon & \text{otherwise.} \end{cases}$$

Since ϵ is small enough, the vector $\nu(\epsilon)$ satisfies the constraints of problem (22). We now define a distribution π_ϵ on the asset price vector x as follows.

$$x = x^\epsilon(i) \text{ with probability } \nu_i(\epsilon),$$

where

$$x_j^\epsilon(i) = \begin{cases} \dfrac{p_j}{\nu_j(\epsilon)} + K_j & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x_j^\epsilon(i)$ is always well-defined, since $\nu_j(\epsilon) > 0$ for every j.

Let us check that the distribution π_ϵ of asset prices satisfies the constraints in (1). For every j, $1 \leq j \leq n$, we have

$$\begin{aligned} \mathbf{E}_{\pi_\epsilon}(x_j - K_j)_+ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(x_j^\epsilon(i) - K_j)_+ \\ &= \nu_j(\epsilon)(x_j^\epsilon(j) - K_j)_+ \\ &= p_j. \end{aligned}$$

We also check that with this choice of asset price distribution, the objective in (1) attains the lower bound $d^{\inf}$, when we let $\epsilon \to 0$. We have

$$\begin{aligned} \mathbf{E}_{\pi_\epsilon}(w^T x - K_0)_+ &= \textstyle\sum_{i=1}^n \nu_i(w^T x^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(\sum_{j=1}^n w_j x_j^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(w_i x_i^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(w_i(\frac{p_i}{\nu_i(\epsilon)} + K_i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n (w_i p_i - \nu_i(\epsilon)(K_0 - w_i K_i))_+. \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\begin{aligned} \lim_{\epsilon \to 0} \mathbf{E}_{\pi_\epsilon}(w^T x - K_0)_+ &= \textstyle\sum_{i=1}^m (w_i p_i - \nu_i(K_0 - w_i K_i))_+ + \sum_{i=m+1}^n w_i p_i \\ &= \textstyle\sum_{i=1}^n (w_i p_i - \nu_i(K_0 - w_i K_i)_+)_+ \\ &= d^{\inf}, \end{aligned}$$

as claimed. This shows that $d^{\inf} = p^{\inf}$ and that the lower bound computed in (3.4) is tight in the absence of constraints on forward prices.

4. Tightness of the integral transform based LP relaxation

In this section, using results from [HLW04] and section 3 we show that the linear programming relaxation derived in Proposition 3 is tight, *i.e.* yields the exact solution to problem (1) in the particular case considered in [HLW04] where only market prices of *single asset* options are given for *many strikes*.

Proposition 6. *Let* $p_i^j \in \mathbf{R}_+$, $K^j \in \mathbf{R}_+$ *for* $j = 1, \ldots, J^i$, $K_0 \in \mathbf{R}_+$ *and* $w_0 \in \mathbf{R}^n_{++}$ *for* $i = 1, \ldots, m$. *The problems*

$$\begin{array}{l} \textit{maximize } \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \textit{subject to } \mathbf{E}_\pi (x_i - K_i^j)_+ = p_i^j, \quad i = 1, \ldots, n, \; j = 1, \ldots, J^i, \end{array} \tag{24}$$

and

$$\begin{array}{ll} \textit{maximize } p_0 & \\ \textit{subject to } \left\langle g_k^i, (w_l^j, K_l^j) - (w_k^i, K_k^i) \right\rangle \leq p_l^j - p_k^i & i, j = 0, \ldots, n, \; k, l = 1, \ldots, J^i \\ \qquad g_{i,i} \geq 0, \quad -1 \leq g_{i,n+1} \leq 0, \quad k = i, \ldots, n & \\ \qquad \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, \ldots, n, \; j = 1, \ldots, J^i & \end{array} \tag{25}$$

where $w_i^j = e_i$ *for* $j = 1, \ldots, J^i$ *and* $J^0 = 2$, *have the same optimal value.*

Proof. We first focus on the particular case where only the forward price and one option price are given per asset, *i.e.* $K_i^1 = 0$ and $J^i = 2$ for $i = 1, \ldots, n$. As in section 3, we write $q_i = p_i^1$ and in order for our problem to be feasible, we assume $0 \leq p < q \leq p + K$.

Let us show that the LP in (25) is feasible. Indeed, we can form a piecewise affine function that is feasible for (7) by taking $C(w_0, K_0) = E_\pi (w_0^T x - K_0)_+$, where π is the probability measure defined in (10). By construction, this function corresponds to a feasible point of (25) and the variables g_i in (25) are simply the subgradients of $C(w_0, K_0)$ at the data points. Finally, the LP in (25) is finite, since we always have $0 \leq E_\pi (w_0^T x - K_0)_+ \leq w_0^T q$ and the feasible set of (25) is compact. This means that the optimum in (25) is attained.

Suppose that the forward price information is ignored, i.e. assuming that $m = n$, and $w_0 \in \mathbf{R}^n_+$. We note e_i, the i-th unit vector. Without loss of generality, we set $w_0^T e = 1$. Since the function $C(w_0, K_0) = w_0^T p + \left(w_0^T K - K_0\right)_+$ is a feasible point of the infinite LP in (7), if we call V^{LP} the upper bound computed by the linear program (25), we must have:

$$V^{\mathrm{LP}} \geq w_0^T p + \left(w_0^T K - K_0\right)_+ .$$

Now, using the necessary conditions in (7) and the convexity of

$$\mathbf{E}_{\pi_\varepsilon} \left(w^T x - K\right)_+$$

in (w,K) we can write

$$\begin{aligned}\mathbf{E}_{\pi_\varepsilon}\left(w_0^Tx-K_0\right)_+ &= \mathbf{E}_{\pi_\varepsilon}\left(w_0^Tx-\left(w_0^TK+\left(K_0-w_0^TK\right)\right)\right)_+\\ &\leq \sum_{i=1}^n w_{0,i}\mathbf{E}_{\pi_\varepsilon}\left(x_i-\left(K_i+\left(K_0-w_0^TK\right)\right)\right)_+\\ &= \sum_{i=1}^n w_{0,i}C\left(e_i,K_i+\left(K_0-w_0^TK\right)\right).\end{aligned}$$

The conditions on the slope of the function $C(w,K)$ imply

$$\sum_{i=1}^n w_{0,i}C\left(e_i,K_i+\left(K_0-w_0^TK\right)\right)\leq w_0^Tp+\left(w_0^TK-K_0\right)_+.$$

Hence, $V^{\mathrm{LP}}\leq w_0^Tp+\left(w_0^TK-K_0\right)_+$ and finally

$$V^{\mathrm{LP}}=w_0^Tp+\left(w_0^TK-K_0\right)_+, \tag{26}$$

where we recover the expression found in (15). This means that the upper bound computed by the LP relaxation is tight in the particular case considered above.

Now we turn to the case when forward price constraints $\mathbf{E}_\pi x_i=q_i$ for $i=1,\ldots,n$, are included. As already observed in 3.1.1, the function

$$d^{\sup}(w_0,K_0)=\max_{0\leq j\leq n+1}: w_0^Tp+\sum_i w_{0,i}\min(q_i-p_i,\beta_jK_i)-\beta_jK_0,$$

is convex in (w_0,K_0). Also, when $w_0=e_i$, and $K_0=K_i$, we obtain $d^{\sup}=p_i$, while for $K_i=0$, we obtain $d^{\sup}=q_i$. This means that $d^{\sup}(w,K)$ is a feasible point of the infinite program (7) and hence $V^{\mathrm{LP}}\geq d^{\sup}(w_0,K_0)$.

Since the finite LP (25) is attained, at a point denoted by

$$z^*=\left[p_0^*,g_0^{*T},\ldots,g_k^{*T}\right]^T,$$

we can define the call price function

$$d^{\mathrm{LP}}(w,K)=\max_{i=0,\ldots,2n+1}\left\{p_i+\langle g_i^*,(w,K)-(w_i,K_i)\rangle\right\},$$

with $p_0=p_0^*$. We can compute the corresponding strike prices $\hat{K}=t^*K$ and option prices $\hat{p}=(1-t^*)q+t^*p$, as in 3.2. By convexity of $d^{\mathrm{LP}}(w,K)$, we have $d^{\mathrm{LP}}(e_i,\hat{K})\leq\hat{p}_i$ for $i=1,\ldots,n$. From (26), we get that $d^{\mathrm{LP}}(w_0,K_0)=V^{\mathrm{LP}}\leq d^{\sup}(w_0,K_0)$, hence finally $d^{\mathrm{LP}}(w_0,K_0)=d^{\sup}(w_0,K_0)$. This shows that the LP relaxation of the upper bound is tight when the input is composed of forwards and one option price per asset.

Now, from section 3.1.2, we know that the upper bound problem given two option prices per asset can be reduced to an upper bound problem given forwards and one option price per asset. This means that the LP relaxation of the upper bound problem is also tight in the case where two option prices are given for each asset. Finally, we conclude by remarking that [HLW04, Th. 4.1] show that the optimal upper bound involves at most two option prices per asset, hence the problem of finding an upper bound in the general problem (24), *i.e.* given option prices for many strikes, reduces to an upper bound problem given only two options per asset, for which the LP relaxation is tight. □

The relaxation we obtain in section 2 treats the upper and lower bound problems in a completely symmetric way. This does not *at all* reflect the inherent complexity of the exact problems. In the upper bound case, we are essentially lucky and we can find, in most cases, discrete distributions with compact support matching the upper bounds, hence we very often get perfect duality (most notably in the special case discussed in this section). This is not true for the lower bound and we almost never get perfect duality in general.

We can also remark that since the largest decreasing convex functions interpolating the market option prices are piecewise affine, the upper bound computed in [HLW04, Theorem 4.1] can also be formulated as the solution to a linear program.

5. Numerical results

In this section, we first show how a variant to the program in Proposition 3 can efficiently "clean" the market data on single asset option prices to remove the non convexities due to noise (asynchronous data, liquidity, etc). We then focus on an equity market example and compare the upper bound computed using the linear programming relaxation in Proposition 6 and the bounds obtained in [HLW04] on the data set described in [HLW04, Table 2]. We show that this same relaxation is not tight in the lower bound case by exhibiting an example where the lower bound computed in section 3.3 is larger than the relaxation's result. Finally, we test our relaxation technique in the general case using a simplified interest rate data set taken from [BM01] and show how the lower and upper bounds perform in the general setting of computing bounds on the price of a basket given other basket prices.

5.1. Basket bounds given single asset option prices

5.1.1. Market data In Figure 1, we show the market price on Mar. 17, 2004 of a certain number of basket call options on the Dow Jones index, with maturity Apr. 16, 2004, for various strikes (the underlying asset of this option is here the Dow Jones index divided by 100). We notice that the convexity requirement with respect to the strike price in Proposition 2 is violated between strikes 95 and 103 for example. Since prices in this plot are last quotes, this does not necessarily mean that an arbitrage is present. In practice, options with strikes far away from the forward price (here equal to 103) tend to be somewhat illiquid and their last quoted price can be an unreliable indication of the price at which they would

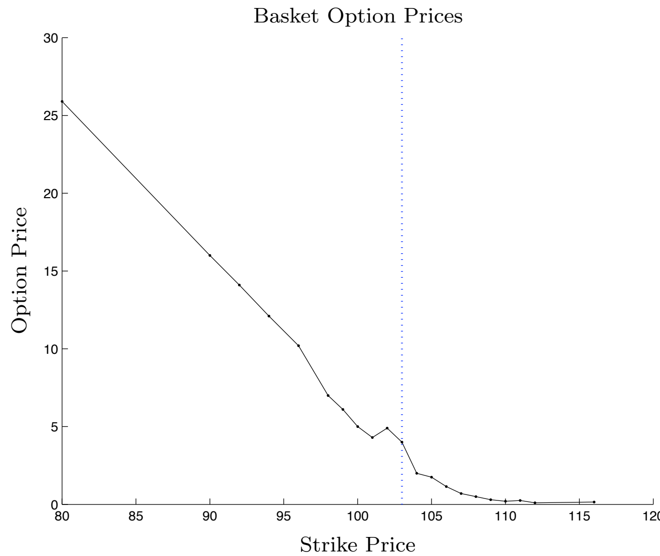

Fig. 1. Dow Jones index call option prices on Mar. 17 2004, with maturity Apr. 16 2004, for various strikes. The vertical dotted line is at the forward price.

trade today. However, since this option price data is used to calibrated pricing models, being able to efficiently detect such anomalies in the data is crucial. We can use the result of Proposition 3 to clean the input data on single asset option prices, from [LL00] we know that the relaxation in Proposition 3 is always exact in dimension one and we look for a decreasing, convex approximation of the market data. Let p_i be the market prices of calls on an asset x with strikes K_i, for $i = 1, \ldots, m$, we can find the closest (in the l_1 norm sense) arbitrage free approximation of the market prices by solving:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} |y_i - p_i| \\ \text{subject to} & \langle g_i, (0, K_j - K_i) \rangle \leq y_j - y_i \\ & g_{i,1} \geq 0, \quad -1 \leq g_{i,2} \leq 0 \\ & \langle g_i, (1, K_i) \rangle = y_i, \quad i, j = 1, \ldots, m, \end{array} \tag{27}$$

which is a linear program in the variables $g_i \in \mathbf{R}^2$ and $y_i \in \mathbf{R}$, for $i = 1, \ldots, m$.

5.1.2. Upper bound with many strikes Here we test empirically the result detailed in Proposition 6 using the data set in [HLW04, Table 2]. This data set still contains minor non convexities (on the BA option prices for example) and we first clean it using the procedure described in (27) above. We then compare the upper bound on the price of a basket option with uniform weights 0.071 obtained using the relaxation in Proposition 3 with those detailed in [HLW04, Table 3]. The results are detailed in Table 1. As expected from the result in Proposition 6, the bounds match up to a small numerical error that is most likely due to differences in the data cleaning procedure.

Strikes	**DJX price**	**LP relax.**	**U.B.**
52	47.10	47.13	47.09
56	43.10	43.14	43.10
60	39.10	39.15	39.11
64	35.10	35.16	35.11
68	31.10	31.17	31.12
70	29.10	29.18	29.13
72	27.10	27.19	27.14
76	23.10	23.21	23.15
80	19.10	19.25	19.18
84	15.20	15.37	15.24
88	11.30	11.63	11.42
90	9.40	9.80	9.61
92	7.50	8.08	7.90
94	5.80	6.49	6.32
95	4.95	5.73	5.57
96	4.15	5.01	4.85
97	3.35	4.33	4.19
98	2.73	3.71	3.58
99	2.13	3.14	3.02
100	1.60	2.64	2.53
102	0.78	1.82	1.73
103	0.50	1.48	1.42
104	0.33	1.22	1.16
105	0.15	1.00	0.95
106	0.15	0.80	0.75
107	0.15	0.64	0.59

Table 1. This table displays for various strikes: the market price (DJX) of a basket option with uniform weights, the upper bound (LP relax.) computed using the relaxation in §2 and the upper bound (U.B.) computed in [HLW04].

Intuitively, tightness in this setting stems from the fact that the convex function $C(w, K)$ which is the optimal solution to the relaxation in Proposition 3 coincides on each axis with the convex envelope of the single asset option prices in the data set (the functions $\bar{C}^{(i)}$ in [HLW04, p. 12]). Unfortunately, this does not happen in the more general case where the data consists of basket option prices.

5.1.3. Lower bound We can also test the tightness of the various bounds obtained above on a simulated data set and compare these bounds with actual prices. The forward prices are given by $q = \{70, 50, 40, 40, 40\}$. We set $K = \{70, 50, 40, 40, 40\}$ and $p = \{1.61, 1.43, 0.93, 0.70, 0.47\}$. Using the results of section 3, we get bounds on the price of a basket option with weight vector $w_0 = \{0.2, 0.2, 0.2, 0.2, 0.2\}$ which are detailed in Table 2.

We notice that the lower bound computed using (19) is tighter than that provided by the LP relaxation in (8) for at least one strike price. In this case, the LP relaxation is equal to the trivial lower bound given by the intrinsic value of the option.

Strike price	3.84	4.32	4.80	5.28	5.76
Upper bound (relax.)	1.71	1.37	1.03	1.03	1.03
Upper bound (from §3)	1.71	1.37	1.03	1.03	1.03
Lower bound (from §3)	0.96	0.48	0.09	0.00	0.00
Lower bound (relax.)	0.96	0.48	0.00	0.00	0.00

Table 2. Lower and upper bounds on the price of a basket given single asset forward prices and one option price per asset. We compare the lower bounds produced in §3 with those produced by the linear programming relaxation in §2.

5.2. Basket bounds given basket option prices

Here, we test the relaxation in Proposition 3 on a simplified data set taken from [BM01]. Our objective here is to look for bounds on the price of a swaption given other swaption prices in a simplified setting. We refer the reader to [Reb98] and [d'A03] for further details.

We compute market option prices by simulation. The underlying asset prices follow lognormal dynamics as in [BS73]. The forwards are given here by $F = \{0.03, 0.04, 0.04, 0.05, 0.05\}$ (these are forward interest rates), and the covariance matrix C is taken from [BM01, pp. 301, 311]:

$$C = \begin{pmatrix} 0.034 & 0.032 & 0.026 & 0.021 & 0.018 \\ 0.032 & 0.035 & 0.019 & 0.026 & 0.011 \\ 0.026 & 0.019 & 0.024 & 0.010 & 0.019 \\ 0.021 & 0.026 & 0.010 & 0.020 & 0.004 \\ 0.018 & 0.011 & 0.019 & 0.004 & 0.017 \end{pmatrix}$$

We look for upper and lower bounds on the price of a basket option (a yield curve option here) with weight vector $w_0 = \{.2, .1, .2, .1, .2\}$ for various strikes, given the simulated price of other at the money basket options (swaptions) with weights vectors given by the lines of the matrix:

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and the at the money price of a basket option with weights w_0 (which is usually liquid). We are looking for bounds on basket options with weight w_0 and strike prices $\pm 10\%$ away from the forward. The resulting price bounds are plotted in Figure 2 in terms of implied volatility. We notice that the market volatility is actually equal (up to a small numerical error) to either the lower bound when the option is in the money or to the upper bound when the option is out of the money.

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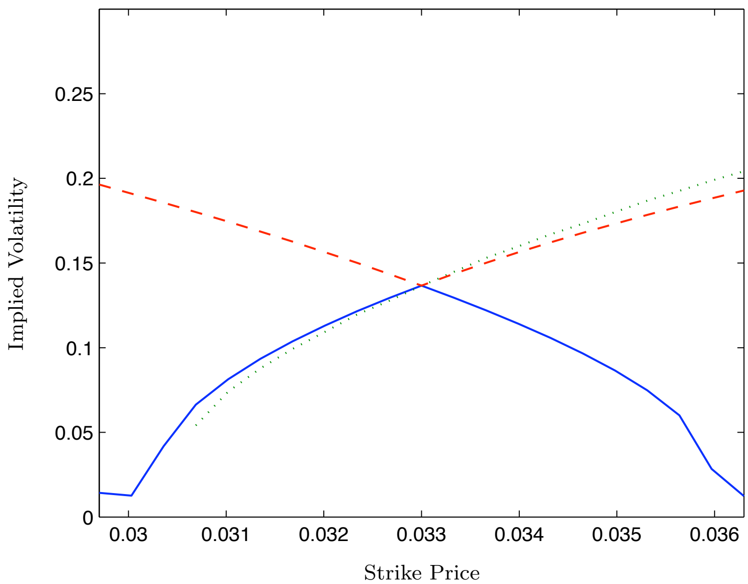

Fig. 2. Upper (dashed) and lower (solid) bounds on the price (plotted here in terms of implied volatility) of a basket option given other basket option prices. These bounds are computed using the linear programming relaxation detailed in §2. The price of the at the money option is given here, hence the upper and lower bounds match at this point. The dotted line is the market volatility.

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