Math221 Homework # 7 Solutions

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Prob. 3.7 The purpose of this question is to illustrate how a decomposition (like QR) that costs $O(n^3)$ can up "updated" at a cost of just $O(n^2)$ when a "small" (i.e. low-rank) change is made to the matrix, eg adding or subtracting a row or column. This applies not just to QR but many of the other factorizations discussed in this book.

The goal of this question is to make $R+uv^T$ upper triangular using just $O(n)$ Givens rotations, each of which costs $O(n)$ to apply, for a total cost of $O(n^2)$. We note that if u were nonzero in its first entry $u(1)$, then we could add $R+uv^T$ and only change the first row of R. So our algorithm will be

1. Premultiply u by $n-1$ Givens rotations to zero out $u(2:n)$. Separately premultiply R by the same Givens rotations, yielding $Q_1^T \cdot (R+uv^T) = Q_1^T R + (Q_1^T u)v$ where Q_1^T is the product of Givens rotations, and $Q_1^T R$ and $Q_1^T u$ are computed separately and not explicitly added (yet). Do this so that $Q_1^T R$ stays "close" to triangular.
2. Add $(Q_1)u^T v$ to the first row of $Q_1^T R$, yielding $Q_1^T(R+u^T v)$.
3. Premultiply $Q_1^T(R+u^T v)$ by $n-1$ more Givens rotations (calling their product Q_2^T) to return it to upper triangular form $\hat{R}$, yielding $Q_2^T Q_1^T(R+uv^T) = \hat{R}$, or $R+uv^T = (Q_1 Q_2)\hat{R} = Q\hat{R}$ as desired.

In more detail:

1. For $i=n$ down to 2, choose a Givens rotation that when multiplied by $[u(i-1); u(i)]$, zeros out the bottom entry. When multiplied by R, this Given rotation will cause entry $R(i,i-1)$ to fill-in. When finished, this means $Q_1^T R$ will have new nonzero entries right below the main diagonal. The cost is bounded by applying $n-1$ Givens rotations to pairs of rows of R of length at most n, for a cost of $O(n^2)$.
2. This costs just n adds.
3. Now $Q_1^T(R+uv^T)$ is nonzero only on and above the first subdiagonal. We zero out the subdiagonal entries one at a time as follows: For $i=1$ to $n-1$, choose a Givens rotation that when multiplied by $[R(i,i); R(i+1,i)]$ zeros out the bottom entry, and multiply it by $Q_1^T(R+uv^T)$.

Accumulating the Givens rotations into an explicit Q matrix costs another $O(n^2)$ flops. See the Matlab code on the webpage for more details.

Prob. 3.17, Part 1 The purpose of this question is to show how to multiply a matrix X by a sequence of k Householder reflections $Q = H_1 \cdots H_k$ where $H_i = I - 2u_i u_i^T$ and $\|u_i\|_2 = 1$, by doing just 3 matrix multiplications, which is generally much faster (because it moves less data). This is worth doing only when X has enough columns, because there is some extra work required.

We will prove that $Q = I - UTU^T$ where $U = [u_1, ..., u_k]$ is n-by-k and T is k-by-k and upper triangular. We use induction. The base case $k = 1$ is simple: $Q = H_1 = I - u_1 \cdot 2 \cdot u_1^T$, so $T = 2$. Assume the result is true for k, and consider postmultiplying $Q = I - UTU^T$ by $H_{k+1} = I - 2u_{k+1}u_{k+1}^T$ to get

$$\begin{aligned} QH_{k+1} &= (I - UTU^T)(I - 2u_{k+1}u_{k+1}^T) \\ &= I - UTU^T - 2u_{k+1}u_{k+1}^T + 2UTU^T u_{k+1}u_{k+1}^T \\ &= I - [U, u_{k+1}] \cdot \begin{bmatrix} T & -2TU^T u_{k+1} \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} U^T \\ u_{k+1}^T \end{bmatrix} \\ &\equiv I - \hat{U}\hat{T}\hat{U}^T \end{aligned}$$

where $\hat{U} = [u_1, ..., u_{k+1}]$ and is n-by-$k+1$ as desired, and $\hat{T} = \begin{bmatrix} T & -2TU^T u_{k+1} \\ 0 & 2 \end{bmatrix}$ is $k+1$-by-$k+1$ and upper triangular as desired. Thus forming $\hat{T}$ from T costs 2 matrix-vector multiplications, $z = U^T u_{k+1}$ and $-2Tz$, for a cost of $O(nk + k^2)$, and so forming T from scratch costs $O(nk^2 + k^3)$.

Prob. 4.1 For simplicty, let $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$. There are (at least) 2 approaches, depending on what more basic linear algebra results that you rely on.

First solution: Here we (mostly) depend on the fact that $\det(AB) = \det(A)\det(B)$. If T_{11} is singular, i.e. $\det(T_{11}) = 0$, then it has a nonzero null vector x, so

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11}x \\ 0 \end{bmatrix} = 0$$

and so T has a null vector and zero determinant as well. Thus $\det(T) = 0 = \det(T_{11}) \cdot \det(T_{22})$.

Otherwise, if T_{11} is nonsingular. then

$$\begin{aligned} \det(T) &= \det(\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}) \cdot 1 \\ &= \det(\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}) \cdot \begin{bmatrix} I & -T_{11}^{-1} \cdot T_{12} \\ 0 & I \end{bmatrix}) \\ &= \det(\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}) \cdot \\ &= \det(\begin{bmatrix} T_{11} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & T_{22} \end{bmatrix}) \\ &= \det(\begin{bmatrix} T_{11} & 0 \\ 0 & I \end{bmatrix}) \cdot \det(\begin{bmatrix} I & 0 \\ 0 & T_{22} \end{bmatrix}) \\ &= \det(T_{11}) \cdot \det(T_{22}) \end{aligned}$$

where the last step follows from the definition of the determinant (as an expansion of minors).

The general result (for more than 2 diagonal blocks) follows by induction.

Subtracting λ from all the diagonal entries does not change the proof.

Second solution: Let $T_{ii} = S_i J_i S_i^{-1}$ be the Jordan form of T_{ii}, so J_i is upper triangular. Then

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} S_1 J_1 S_1^{-1} & T_{12} \\ 0 & S_2 J_2 S_2^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \cdot \begin{bmatrix} J_1 & S_1^{-1} T_{12} S_2 \\ 0 & J_2 \end{bmatrix} \cdot \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}^{-1}$$

is a similarity transformation, so T and $\begin{bmatrix} J_1 & S_1^{-1} T_{12} S_2 \\ 0 & J_2 \end{bmatrix}$ have the same eigenvalues. But the latter matrix is upper triangular, so its eigenvalues are on its diagonal, namely the union of the eigenvalues of T_{11} and T_{22}.

The general result (for more than 2 diagonal blocks) follows by induction.

Prob. 4.2 We again proceed by induction. Let $A = \begin{bmatrix} U & x \\ 0 & \alpha \end{bmatrix}$ be upper triangular, where $\alpha = A_{nn}$ is a scalar. Then

$$(A^* \cdot A)_{nn} = x^* \cdot x + \bar{\alpha}\alpha = \|x\|_2^2 + |\alpha|^2$$

and

$$(A \cdot A^*)_{nn} = \alpha\bar{\alpha} = |\alpha|^2$$

must be equal, i.e. $x = 0$. Then we see that $A \cdot A^* = A^* \cdot A$ implies

$$\begin{bmatrix} U \cdot U^* & 0 \\ 0 & |\alpha|^2 \end{bmatrix} = \begin{bmatrix} U^* \cdot U & 0 \\ 0 & |\alpha|^2 \end{bmatrix}$$

so by induction U is diagonal so A is diagonal as desired.

Now if Q is any unitary matrix, then $(QAQ^*)^*(QAQ^*) = QA^*AQ^*$ and $(QAQ^*)(QAQ^*)^* = QAA^*Q^*$. thus $(QAQ^*)^*(QAQ^*) = (QAQ^*)(QAQ^*)^*$ if and only if $A^*A = AA^*$, ie. A is normal if and only if QAQ^* is normal. Choosing Q so that QAQ^* is in Schur form, i.e. upper triangular, we see that A is normal if and only if its Schur form is normal, which is true if and only if its Schur form is a diagonal matrix Λ, i.e. $A = Q^*\Lambda Q$, i.e. A has eigenvalues Λ_{ii} with orthogonal eigenvectors (the columns of Q^*).

Prob. 4.3 $y^* \cdot A \cdot x = y^* \cdot (A \cdot x) = y^* \cdot (\lambda x) = \lambda(y^* \cdot x)$ and $y^* \cdot A \cdot x = (y^* \cdot A) \cdot x = (\mu y^*) \cdot x = \mu(y^* \cdot x)$. Thus $\lambda(y^* \cdot x) = \mu(y^* \cdot x)$ and $\lambda \neq \mu$; this can only be true if $y^* \cdot x = 0$, i.e. y and x are orthogonal.

Prob. 4.4 Part 1 If $A = QTQ^*$ is the Schur form of A then $A^n = (QTQ^*)^n = (QTQ^*)(QTQ^*) \cdots (QTQ^*) = QT^nQ^*$, since all the intermediate Q^*s and Qs cancel. Thus $f(A) = \sum_i a_i A^i = \sum_i a_i Q T^i Q^* = Q(\sum_i a_i T^i)Q^* = Qf(T)Q^*$.

Part 2 If U and $\hat{U}$ are upper triangular, then $(U\hat{U})_{jj} = \sum_k U_{jk}\hat{U}_{kj}$; the only nonzero contribution to this sum is $U_{jj}\hat{U}_{jj}$, i.e. the diagonal entries just multiply. By induction $(T^n)_{jj} = (T_{jj})^n$, and

$$(f(T))_{jj} = (\sum_i a_i T^i)_{jj} = \sum_i a_i (T^i)_{jj} = \sum_i a_i (T_{jj})^i = f(T_{jj})$$

Part 3

$$Tf(T) = T(\sum_i a_i T^i) = \sum_i a_i T^{i+1} = (\sum_i a_i T^i)T = f(T)T$$

Part 4 Since T is upper triangular, so are T^i and $f(T)$. Equating the i, j entries of $Tf(T)$ and $f(T)T$, where $i < j$, we get

$$\sum_{k=i}^{j} T_{ik}(f(T))_{kj} = \sum_{k=i}^{j} (f(T))_{ik} T_{kj}$$

or

$$T_{ii}(f(T))_{ij} + \sum_{k=i+1}^{j} T_{ij}(f(T))_{kj} = \sum_{k=i}^{j-1} (f(T))_{ik} T_{kj} + T_{jj}(f(T))_{ij}$$

or

$$(f(T))_{ij}(T_{ii} - T_{jj}) = \sum_{k=i}^{j-1} (f(T))_{ik} T_{kj} - \sum_{k=i+1}^{j} T_{ij}(f(T))_{kj}$$

If A has distinct eigenvalues, we can divide by $T_{ii} - T_{jj}$ to get the formula

$$(f(T))_{ij} = (\sum_{k=i}^{j-1} (f(T))_{ik} T_{kj} - \sum_{k=i+1}^{j} T_{ij}(f(T))_{kj})/(T_{ii} - T_{jj})$$

Since $(f(T))_{ij}$ lies on the $j-i$-th superdiagonal, this formula expresses the entries of $f(T)$ on the $j-i$-th superdiagonal in terms of entries of $f(T)$ on the $k-i$-th superdiagonal (with $k < j$) and on the $j-k$-th superdiagonal (with $k > i$), i.e. on lower numbered superdiagonals, i.e. those close to the main diagonal. So by computing the main diagonal $f(T)_{ii} = f(T_{ii})$ for all i, then the first superdiagonal $f(T)_{i,i+1}$ for all i, then the second superdiagonal $f(T)_{i,i+2}$ for all i, and so on, the formula above can be used to compute all the entries of $f(T)$, and so of $f(A) = Qf(T)Q^*$.

Prob. 4.5 If $A = QTQ^*$ is the Schur canonical form of A, with T upper triangular, then A and T have the same eigenvalues, namely T_{ii}. From the last question $f(A) = Qf(T)Q^*$ is the Schur canonical form of $f(A)$, so $f(A)$ and $f(T)$ have the same eigenvalues, namely $(f(T))_{ii} = f(T_{ii})$.