CS 70 SPRING 2008 — DISCUSSION #8

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Let $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

1. Balls and Bins

Balls and bins problems, although dry and seem like military drills, form a great model for basic combinatorics problems (and also for randomized hashing). The premise is that we would like to throw m balls into n bins and each bin can fit multiple balls. We distinguish one final configuration from another only by the content of the bins, not by the order in which the balls land in the bins.

Exercise 1. First, suppose that the m balls and the n bins are *distinguishable*, i.e. each balls is labeled with a different number and each bin is given a distinct name. How many ways are there to throw the m balls into the n bins then?

Exercise 2. Suppose that the balls and bins are still distinguishable, that $n > m$, and that we would like each bin to have at most 1 ball. How many ways are there to throw the balls?

Exercise 3. We still have distinguishable m balls and n bins; how many ways can we throw the balls if we want the first and the second bins to have exactly two balls?

Exercise 4. Same premise as the above problem, but now $m = 2n$ and we would like every bin to have exactly 2 balls. Min immediately tells the following solution to Aaron:

> We first place the m balls in a line. Then we can throw the 1st and 2nd balls into the first bin, the 3rd and 4th balls into the second bin, etc and satisfy the problem condition. Thus, the number of ways to throw the balls is just the number of ways to arrange m distinct balls in a line, which is just $m!$.

First, explain why Aaron is in complete dismay upon hearing this 'solution'.

Next, find the correct answer to this problem.

Exercise 5. We can generalize the previous problem. Suppose we have $r_1 + r_2 + ...r_n = m$ where r_i are all natural numbers. We would like the i-th bin to have r_i number of balls, how many ways are there to throw the balls?

Exercise 6. We still have distinguishable m balls and n bins where $m > n$ and we would like to find the number of ways to throw the balls if we want every bin to have at least one ball. A Stanford student proposes the following calculation:

> We first choose n balls from the m total and put one in each bin. Since orders matter because the balls are distinguishable, there are $m!/(m-n)!$ ways to do this. Now, after we distribute the first batch of n balls, we have to throw remaining $m-n$ balls into n bins. This second step has n^{m-n} possible choices, hence, the total is $(m!/(m-n)!)n^{m-n}$.

Why would you not put this answer on the test?

Another Stanford student offers an alternative strategy:

> We first figure out how many ways there are to throw the balls such that at least one bin is empty. There are clearly n ways to choose the bin that we will force to remain empty. After we have chosen the lonely bin, we have $(n-1)^m$ ways to distribute m balls into the remaining $n-1$ bins. Hence, there are $n(n-1)^m$ ways to throw the balls such that at least one bin is empty. Now, there are n^m total ways to throw m balls into n bins, and each throwing scheme either leave at least one bin empty or puts at least one ball in every bin, so there are then $n^m - n(n-1)^m$ ways to have at least one ball in every bin.

Date: March 18, 2008.

Is this answer better? Why or why not?

Exercise 7. In the previous balls and bins problems, we did not care about the order in which the balls land in any given bin. But now, suppose that the bins have a sensor that can tell which ball was thrown into it first so that throwing ball A and then ball B into bin 3 is now considered a different configuration from throwing ball B and then ball A into bin 3. The same Stanford student from the previous problem quickly formulates the following solution:

> We already know that if we throw the balls in alphabetical order, i.e., we throw ball A, and then ball B, and then C, etc, then we have n^m possible ways to distribute the balls. But in this scheme, if ball A and ball D land in the same bin, ball A would always land in the bin first. To count the configuration where ball D land before ball A in the bin, we can just change the order in which we throw the balls, i.e., we can always throw ball D before ball A. Hence, for each different orders we throw the balls, we can get a n^m possible distributions. Since there are $m!$ different orders to throw the balls, there are $m!n^m$ total configurations.

Do you think the Stanford student is right this time? Why or why not?

Exercise 8. Now, suppose that the balls are identical and hence *indistinguishable*, i.e. having ball A in bin 1 and ball B in bin 3 will now be considered the same as having ball A in bin 3 and ball B in bin 1. We would like to find the number of ways to throw the balls with this new condition. Against your better judgement, you decide to, again, consult a Stanford student who offers you the following solution:

> Recall that when the 2 'E's in 'CHEESY' are distinguishable, we first assume that they are different and get 6! ways to re-arrange the letters and then divide by 2! because we can permute the order of the 'E's. We can use similar strategy here. We first assume that the balls are distinguishable. Then we already know that there are n^m ways to throw the balls. Now, since the balls are indistinguishable, we can look at the final configuration, permute the balls in the bins and still get the the same configuration (i.e. if in one configuration, ball A is in bin 1, ball B is in bin 2, we can permute the order so that ball B is in bin 1 and ball A is in bin 2 and get the same configuration). Thus, there are then $n^m/m!$ ways to throw the balls.

Explain where this approach went wrong.

Now, formulate your own solution to this problem.

Exercise 9. Suppose we have 5 distinguishable balls and 5 *indistinguishable* bins. How many ways are there to throw the balls?

Exercise 10. Same premise as the above problem but now the balls are also *indistinguishable*. How many ways are there to throw the balls?

2. Remembering Power Expansion Without Tears

Exercise 11. Use combinatorics to prove that $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

3. Pigeonhole Principle

Exercise 12. There are six people at a party. Each pair of people are either friends or enemies. Prove that there are three people at the party who are all friends or who are all enemies.

Exercise 13. Given a sequence $a_1, a_2, \ldots, a_{2008}$ of 2008 integers, show there is a consecutive subsequence whose sum is divisible by 2008. (A consecutive subsequence of is a a subsequence of the form $a_i, a_{i+1}, \ldots, a_{i+n-1}$ for some $n \geq 1$).