CS 70 SPRING 2008 — DISCUSSION #11

LUQMAN HODGKINSON, AARON KLEINMAN, MIN XU

1. Random Variables

We say that X, Y are independent random variables if $\forall m, n \in \mathbb{Z}. \Pr[X(n) \cap Y(m)] = \Pr[X(n)] \Pr[Y(m)]$

Exercise 1. Let X, Y be independent random variables. Answer the following T/F questions:

(1) if $\Pr[X > Y] = 1$, then $\mathbb{E}[X] > \mathbb{E}[Y]$
(2) If X, Y are constant, then XY is constant also
(3) $\Pr[X > \mathbb{E}[X]] = \Pr[X < \mathbb{E}[X]]$
(4) if $\mathbb{E}[X] > \mathbb{E}[Y]$, then $\Pr[X > Y] > \Pr[X < Y]$

2. Calculating Expectation

Exercise 2. A casino offers the following game: it pays you \$14. Then you roll a die, square that number, and pay the casino that amount of dollars. Your friend claims that this game is profitible for you, and reasons as follows. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of possible outcomes of the die with uniform probability, let X be the random variable which is the amount of money that you must pay the casino, and let Y be the random variable which is the number rolled. For example, $X(2) = 4$ and $Y(2) = 2$. Then $X = Y^2$, so $\mathbb{E}[X] = \mathbb{E}[Y^2]$. By linearity of expectation, or so your friend claims, $\mathbb{E}[Y^2] = (\mathbb{E}[Y])^2$, so

$$\mathbb{E}[X] = (\mathbb{E}[Y])^2 = (3.5)^2 = 12.25$$

Thus the average amount that you must pay the casino is \$12.25. Since this is less than the \$14 that the casino pays you to play, the game is profitible for you.

Is your friend correct? Why or why not?

3. Indicator Random Variables and Linearity of Expectation

Exercise 3. You are dealt 13 cards from a standard shuffled deck. What is the expected number of aces that you receive?

Exercise 4. A monkey types on a 26-letter keyboard, with all lowercase letters. Assume that the monkey choses each character independently and uniformly at random, and that it types $1,000,000,000$ characters total. What is the expected number of times the sequence "hamlet" appears?

Exercise 5. Suppose that Aaron chooses a permutation of the numbers 1, 2, ..., n uniformly at random. What is the expected number of entries that are greater than all preceding entries? For example, in the permutation 4, 2, 1, 5, 3, the numbers 4 and 5 are greater than all preceding entries (Hint: Whats the probability that the first entry is greater than all the preceding entries? What about the second one?)

Date: April 17, 2008.