Counting

In the next major topic of the course, we will be looking at probability. Suppose you toss a fair coin a thousand times. How likely is it that you get exactly 500 heads? And what about 1000 heads? It turns out that the chances of 500 heads are roughly 5%, whereas the chances of 1000 heads are so infinitesimally small that we may as well say that it is impossible. But before you can learn to compute or estimate odds or probabilities you must learn to count! That is the subject of this note.

We will learn how to count the number of outcomes while tossing coins, rolling dice and dealing cards. Many of the questions we will be interested in can be cast in the following simple framework called the occupancy model:

Balls in Bins: We have a set of k balls. We wish to place them into n bins. How many different possible outcomes are there?
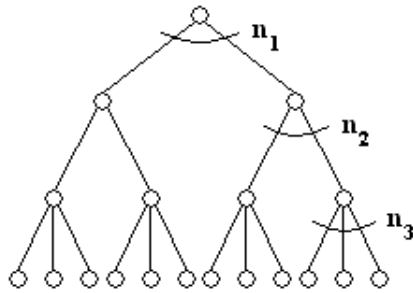
How do we represent coin tossing and card dealing in this framework? Consider the case of $n = 2$ bins labelled H and T, corresponding to the two outcomes of a coin toss. The placement of the k balls correspond to the outcomes of k successive coin tosses. To model card dealing, consider the situation with 52 bins corresponding to a deck of cards. Here the balls correspond to successive cards in a deal.

The two examples illustrate two different constraints on ball placements. In the coin tossing case, different balls can be placed in the same bin. This is called *sampling with replacement*. In the cards case, no bin can contain more than one ball (i.e., the same card cannot be dealt twice). This is called *sampling without replacement*. As an exercise, what are n and k for rolling dice? Is it sampling with or without replacement?

We are interested in counting the number of ways of placing k balls in n bins in each of these scenarios. This is easy to do by applying the first rule of counting:

First Rule of Counting: If an object can be made by a succession of choices, where there are n_1 ways of making the first choice, and *for every* way of making the first choice there are n_2 ways of making the second choice, and *for every* way of making the first and second choice there are n_3 ways of making the third choice, and so on up to the n_k-th choice, then the total number of distinct objects that can be made in this way is the product $n_1 \cdot n_2 \cdot n_3 \cdots n_k$.

Here is another way of picturing this rule: consider a tree with branching factor n_1 at the root, n_2 at every node at the second level, ..., n_k at every node at the k-th level. Then the number of leaves in the tree is the product $n_1 \times n_2 \times n_3 \times \cdots \times n_k$. For example, if $n_1 = 2$, $n_2 = 2$, and $n_3 = 3$, then there are 12 leaves (i.e., outcomes):
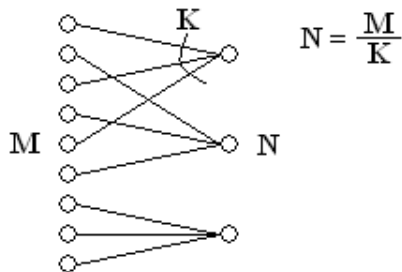
Let us apply this counting rule to figuring out the number of ways of placing k balls in n bins with replacement, if the balls are numbered from 1 to k. This is easy; it is just n^k: n choices for the first ball, n for the second, and so on.

The rule is more interesting in the case of sampling without replacement. Now there are n ways of placing the first ball, and *no matter* where it is placed there are exactly $n-1$ bins in which the second ball may be placed (exactly which $n-1$ depends upon which bin the first ball was placed in, but the number of choices is always $n-1$), and so on. So as long as $k \leq n$, the number of placements is $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$. By convention we assume that $0! = 1$.

Counting Unordered Sets

When dealing a hand of cards, say a poker hand, it is more natural to count the number of distinct hands, without regard to the order in which the cards in our hand were dealt to us. To count this number we use the second rule of counting:

Second Rule of Counting: If an object is made by a succession of choices, and the order in which the choices is made does not matter, count the number of ordered objects (pretending that the order matters), and divide by the number of ordered objects per unordered object. Note that this rule can only be applied if the number of ordered objects is the same for every unordered object.
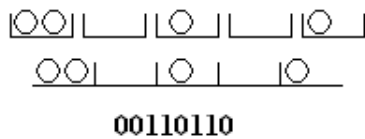

Let us continue with our example of a poker hand. We wish to calculate the number of ways of choosing 5 cards out of a deck of 52 cards. So we first count the number of ways of dealing a 5 card hand pretending that we care which order the cards are dealt in. This is exactly $\frac{52!}{47!}$ as we computed above. Now we ask: for a given poker hand, how many ways could it have been dealt? The 5 cards in the given hand could have been dealt in any one of 5! ways. Therefore by the second rule of counting, the number of poker hands is $\frac{52!}{47! \times 5!}$.

This quantity $\frac{n!}{(n-k)!k!}$ is used so often that there is special notation for it: $\binom{n}{k}$, pronounced *n choose k*. This is the number of ways of placing k balls in n bins (without replacement), where the order of placement does not matter.

What about the case of sampling with replacement? How many ways are there of placing k balls in n bins with replacement when the order does not matter? Let us try to use the second rule of counting. There are
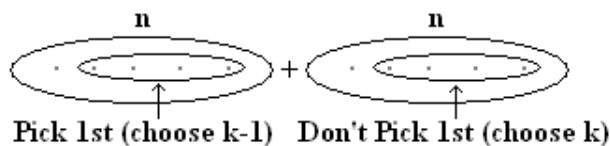
n^k ordered placements. How many ordered placements are there per unordered placement? Unfortunately this depends on which unordered placement we are considering. For example, when $k = 2$ (two balls), if the two balls are in distinct bins then there are two corresponding ordered placements, while if they are in the same bin then there is just one corresponding ordered placement. Thus we have to consider these two cases separately. In the first case, there are n ways to place the first ball, and $n-1$ ways to place the second ball, giving us $n(n-1)$ corresponding ordered placements; by the second rule of counting, we divide by 2 and get $\frac{n(n-1)}{2}$ unordered placements of the balls in distinct bins. In the second case, there are n ways to place both balls in the same bin; by the second rule of counting, we divide by 1 and get n unordered placements for the balls in the same bin. Putting both cases together, there are $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ ways to place two balls into n bins where order does not matter. For larger values of k, it seems hopelessly complicated. Yet there is a remarkably slick way of calculating this number. Represent each of the balls by a 0 and the separations between boxes by 1's. So we have k 0's and $(n-1)$ 1's. Each placement of the k balls in the n boxes corresponds uniquely to a binary string with k 0's and $(n-1)$ 1's. Here is a sample placement of $k = 4$ balls into $n = 5$ bins and how it can be represented as a binary string:

$$00110110$$

But the number of such binary strings is easy to count: we have $n + k - 1$ positions, and we must choose which k of them contain 0's. So the answer is $\binom{n+k-1}{k}$.

Combinatorial Proofs

Combinatorial arguments are interesting because they rely on intuitive counting arguments rather than algebraic manipulation. For example, it is true that $\binom{n}{k} = \binom{n}{n-k}$. Though you may be able to prove this fact rigorously by definition of $\binom{n}{k}$ and algebraic manipulation, some proofs are actually much more tedious and difficult. Instead, we will try to discuss what each term means, and then see why the two sides are equal. When we write $\binom{n}{k}$, we are really counting how many ways we can choose k objects from n objects. But each time we choose any k objects, we must also leave behind $n-k$ objects, which is the same as choosing $n-k$ (to leave behind). Thus, $\binom{n}{k} = \binom{n}{n-k}$. Some facts are less trivial. For example, it is true that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. The two terms on the right hand side are splitting up choosing k from n objects into two cases: we either choose the first element, or we do not. To count the number of ways where we choose the first element, we have $k-1$ objects left to choose, and only $n-1$ objects to choose from, and hence $\binom{n-1}{k-1}$ ways. For the number of ways where we don't choose the first element, we have to pick k objects from $n-1$ this time, giving $\binom{n-1}{k}$ ways. [Exercise: Check algebraically that the above formula holds.]

We can also prove even more complex facts, such as $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$. What does the right hand side really say? It is splitting up the process into cases according to the first (i.e., lowest-numbered)

object we select. In other words:

$$\textbf{First element selected is either} \begin{cases} \text{element 1,} & \binom{n-1}{k} \\ \text{element 2,} & \binom{n-2}{k} \\ \text{element 3,} & \binom{n-3}{k} \\ \quad\vdots \\ \text{element}(n-k), & \binom{k}{k} \end{cases}$$

(Note that the lowest-numbered object we select cannot be higher than $n-k$ as we have to select k distinct objects.)

The last combinatorial proof we will do is the following: $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. To see this, imagine that we have a set S with n elements. On the left hand side, the ith term counts the number of ways of choosing a subset of S of size exactly i; so the sum on the left hand side counts the total number of subsets (of any size) of S. But we claim that the right hand side (2^n) does indeed count the total number of subsets. To see this, just identify a subset with an n-bit vector, where in each position j we put a 1 if the jth element is in the subset, and a 0 otherwise. So the number of subsets is equal to the number of n-bit vectors, which is 2^n. Let us look at an example, where $S = \{1,2,3\}$ (so $n = 3$). Enumerate all $2^3 = 8$ possible subsets of S: $\{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. The term $\binom{3}{0}$ counts the number of ways to choose a subset of S with 0 elements; there is only one such subset, namely the empty set. There are $\binom{3}{1} = 3$ ways of choosing a subset with 1 element, $\binom{3}{2} = 3$ ways of choosing a subset with 2 elements, and $\binom{3}{3} = 1$ way of choosing a subset with 3 elements (namely, the subset consisting of the whole of S). Summing, we get $1+3+3+1 = 8$, as expected.