Due Thursday, March 20th

1. (16 pts.) Practice

The only way to learn counting is to practice, practice, practice—so here is your chance to do so. No need to justify your answers or show your calculations on this problem. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many 10-bit strings are there that contain exactly 4 ones?

(b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

(c) How many different 13-card bridge hands are there that contain no aces?

(d) How many different 13-card bridge hands are there that contain all four aces?

(e) How many different 13-card bridge hands are there that contain exactly 6 spades?

(f) How many 99-bit strings are there that contain more ones than zeros?

(g) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

(h) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(i) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

(j) How many different anagrams of ALASKA are there?

(k) How many different anagrams of ALABAMA are there?

(l) How many different anagrams of MONTANA are there?

(m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins?

(n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty?

(o) How many different ways are there to throw 9 identical balls into 27 bins?

(p) There are exactly 130 students currently enrolled in CS70. How many different ways are there to pair up the 130 CS70 students, so that each student is paired with one other student?

2. (6 pts.) Bijections

(a) Let S denote the set of 24-bit strings that have exactly seven 1's. Let T denote the set of 30-bit strings that have exactly seven 1's but where no two 1's are consecutive. Describe a bijection $f : S \to T$.

(b) We have 30 empty bird boxes all lined up in a row. We have seven bluebirds, but these birds are pretty anti-social: no bluebird is willing to nest in a bird box that's adjacent to another occupied bird box. How many ways are there to place the seven bluebirds into some seven of these bird boxes, so that no two blue birds end up in adjacent bird boxes?

3. (5 pts.) Exam scheduling

The Dean of Students has issued a new policy: No student should have to take two engineering final exams on the same day—that's just cruel. Now it's up to the Dean's staff to select each engineering course's final exam date, in a way that complies with this policy. And to prevent the final exam period from dragging out, the Dean would like to be parsimonious and minimize the number of days in the final exam period.

An alert alumni of CS70 comes up with the following strategy. Form an undirected graph G with one vertex for each engineering course. Draw an edge between two engineering courses if there is some student who is taking both courses. Suppose that we find a coloring of this graph using k colors. Explain how the Dean's staff can use this coloring to assign an exam date to each course, using only k different dates.

4. (8 pts.) Bipartite graphs

Let G be a bipartite graph, with a set L of vertices on the left and a set R of vertices on the right and a non-empty set of edges $E \subseteq L \times R$.

(a) Prove that $\sum_{v \in L} \text{degree}(v) = \sum_{v \in R} \text{degree}(v)$.

(b) Let s denote the average degree of vertices in L and t the average degree of vertices in R, i.e.,

$$s = \frac{1}{|L|} \sum_{v \in L} \text{degree}(v), \quad t = \frac{1}{|R|} \sum_{v \in R} \text{degree}(v).$$

Prove that $s/t = |R|/|L|$.

(c) In 1992, the University of Chicago interviewed a random sample of 2500 people in the U.S. about the number of opposite-gender sex partners they've had. They reported that on average men have 74% more opposite-gender partners than women. At around the same time, the U.S. Census Bureau reported that female population of the U.S. was about 140 million and the male population about 134 million. Explain why the University of Chicago and the U.S. Census Bureau can't both be right.

5. (11 pts.) Fermat's necklace

In the following parts, let p be a prime number and let k be a positive integer.

(a) We have an endless supply of beads. The beads come in k different colors. All beads of the same color are identical and indistinguishable. We have a piece of string. We want to make a pretty decoration by threading p beads onto the string (from left to right). We can choose any sequence of colors, subject only to one rule: the p beads must not all have the same color.

How many different ways are there construct such a sequence of beads?

(Your answer should be a simple function of k and p.)

(b) Now we tie the two ends of the string together, forming a circular necklace. This lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

Count how many non-equivalent necklaces there are, if the p beads must not all have the same color.

(Your answer should be a simple function of k and p.)

Revised 3/17/2008 to fix a typo in the example of how cyclic shifts work.

(c) Use your answer to part (b) to prove Fermat's little theorem. (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)

6. (4 pts.) Grade this proof

Assign a grade of A (correct) or F (failure) to the following proof. If you give a F, please explain clearly where the logical error in the proof lies. Saying that the claim is false is *not* a valid explanation of what is wrong with the proof. If you give an A, you do not need to explain your grade.

Theorem: Let $G = (V, E)$ be a connected, undirected graph and $f : V \to \mathbb{R}$ be a function that labels each vertex with a real number. Suppose that for every vertex $v \in V$, $f(v)$ is the median of the values of f on the vertices adjacent to v (choosing the smaller of the two possible medians, if there are an even number of vertices adjacent to v)—in which case we say that f is *unwrinkled*. Then f is a constant function, i.e., f assigns the same real number to every vertex.

Proof: We use induction on the number of vertices. Let $P(n)$ be the proposition that the theorem is true for every graph with n vertices.

Base case: If $n = 1$, then the graph has only one vertex, so the function f obviously assigns the same real number to every vertex. Therefore $P(1)$ is true.

Induction hypothesis: Assume $P(n)$ is true for some $n \geq 1$.

Inductive step: We must prove that $P(n+1)$ is true. Consider any n-vertex graph G that is connected and any unwrinkled function $f : V \to \mathbb{R}$. By the induction hypothesis, f is a constant function, say $f(v) = c$ for every $v \in V$.

Now we add one more vertex w to obtain a graph $G' = (V', E')$ with $n+1$ vertices, and we extend f to a function $f' : V' \to \mathbb{R}$ by labeling w with some real number. (Here $V' = V \cup \{w\}$, $f'(v) = f(v)$ for all $v \in V$, and E' differs from E only in that it may contain some extra edges incident on w.) Suppose G' is connected and f' is unwrinkled (otherwise there is nothing to prove). Since G' is connected, w must have an edge to at least one vertex in V (possibly more), say to the vertices $v_1, \ldots, v_k$. Since f' is unwrinkled, $f'(w)$ must be equal to the median of the values $f'(v_1), \ldots, f'(v_k)$ (choosing the smaller of the two possible medians in case k is even), which in turn is equal to the median of the values $f(v_1), \ldots, f(v_k)$ (again, choosing the smaller of the two possible medians if k is even). But we said earlier that $f(v) = c$ for every $v \in V$, so $f'(w)$ must be the median of $c, c, \ldots, c$. This means that $f'(w) = c$. Also we know that $f'(v) = f(v) = c$ for every $v \in V$. Therefore $f'(x) = c$ for every $x \in V'$, i.e., f' is a constant function. This proves that for every connected graph G' with $n+1$ vertices and every unwrinkled function f' on G', f' is a constant function. In other words, $P(n+1)$ is true. The theorem follows by induction. $\square$