CS 70 Spring 2008 — Discrete Mathematics for CS, David Wagner — HW 4

Due Thursday, February 21st

1. (25 pts.) Big-O notation

The purpose of this problem is to teach you Big-O notation in a careful way. First, study the following.

Formally: If $f(n), g(n)$ are two non-negative functions of a single integer variable, the statement $f(n) \in \mathrm{O}(g(n))$ means that

$$\exists N_0 \in \mathbb{N} \,.\, \exists C \in \mathbb{N} \,.\, \forall x \in \mathbb{N} \,.\, (x \geq N_0 \implies 0 \leq f(x) \leq C \cdot g(x)).$$

In other words, $\mathrm{O}(g(n))$ is the set of functions $\{f_i(n) : \exists N_0 \in \mathbb{N}.\ \exists C \in \mathbb{N}.\ \forall x \in \mathbb{N}.\ x \geq N_0 \implies f_i(x) \leq C \cdot g(x)\}$. This is the definition of Big-O notation.

Informally: $f(n) \in \mathrm{O}(g(n))$ means, roughly, that $f(n)$ grows "no faster than" $g(n)$ (except possibly for a constant factor), as n gets large. For instance, $n^2 \in \mathrm{O}(n^2)$, $n(n+1)/2 \in \mathrm{O}(n^2)$, and $10000n^2 \in \mathrm{O}(n^2)$, because these functions all grow at asymptotically the same rate (ignoring constant factors). Also, $n^2 \in \mathrm{O}(n^3)$, because n^2 grows more slowly than n^3 does, as n gets large.

Some basic facts: If $f(n) \in \mathrm{O}(g(n))$ and $f'(n) \in \mathrm{O}(g'(n))$, then $f(n) + f'(n) \in \mathrm{O}(g(n) + g'(n))$. If $f(n) \in \mathrm{O}(g(n))$ and $f'(n) \in \mathrm{O}(g'(n))$, then $f(n) \times f'(n) \in \mathrm{O}(g(n) \times g'(n))$.

Common notation: Instead of writing $f(n) \in \mathrm{O}(g(n))$, almost everyone instead writes $f(n) = \mathrm{O}(g(n))$. Strictly speaking, this is a sloppy abuse of notation, but this practice is widespread; you are guaranteed to see it throughout your studies of computer science, so be prepared. Also, we often write something like n^2 as a shorthand for the function $f(n) = n^2$, just to make our life easier.

Now, with that background established, do the following problems:

1. Prove that $n^2 + 2008 \in \mathrm{O}(n^3)$.
   Hint: One possible approach is to give an example of constants N_0, C that satisfy the definition.
2. Prove that $77n^3 \lg n \in \mathrm{O}(n^4)$.
3. True or false: There exists $e \in \mathbb{N}$ such that $2^n \in \mathrm{O}(n^e)$. Briefly justify your answer.
4. Prove that if $f(n) \in \mathrm{O}(g(n))$ and $g(n) \in \mathrm{O}(h(n))$, then $f(n) \in \mathrm{O}(h(n))$.
5. Critique the following argument. Is the reasoning valid? If not, why not? If there is an error, identify the erroneous step and explain what's wrong with it.

   We have $n^2 = \mathrm{O}(n^4)$.
   Also, we have $n^2 = \mathrm{O}(n^3)$.
   By transitivity, it follows that $\mathrm{O}(n^4) = \mathrm{O}(n^3)$.
   This means that $n^4 = \mathrm{O}(n^3)$.

2. (25 pts.) Another pairing roblem

In this problem, we will study a matching problem that is somewhat different from the stable marriage problem. As before, we have n men and n women, but there are no preference lists. Instead, we have a list of potential couples (m,w) who are considered "compatible" (where m is a man and w a woman). The task is to find a pairing of all the men and women, subject to the restriction that a man and woman can only be paired together if they are compatible.

In other words, we are given a compatibility set C, where $(m,w) \in C$ means that (m,w) are a compatible couple. We say that a pairing is a *compatible pairing* if it is made solely out of compatible couples from C. Given C, the goal is to find a compatible pairing. (Any such pairing will do. Since we don't have preference lists, we don't have to worry about stability.)

If S is a set of some number of men, let $f(S)$ denote the set of women who are compatible with some man in S, i.e., $f(S) = \{w : \exists m \in S.(m,w) \in C\}$. We'll say that the compatibility set C is *plentiful* if for every set S of at most n men, $|f(S)| \geq |S|$. (Recall that $|S|$ denotes the *size* of the set S, i.e., the number of elements it has.) We'll say that the compatibility set C is *super-plentiful* if for every non-empty set S of at most $n-1$ men, $|f(S)| > |S|$. In this problem, you will prove that a compatible pairing exists if and only if C is plentiful. (There exist efficient algorithms to find such a pairing, if it exists, but they are too complex to cover here.)

1. Not every C has a compatible pairing. Prove that if C has a compatible pairing, then C is plentiful.
2. Suppose that C is plentiful. Let (a,b) be a compatible couple from C, i.e., $(a,b) \in C$. Suppose that a and b fall madly in love, elope, and fly off to Hawaii for their honeymoon. We are left with $n-1$ men, $n-1$ women, and the compatibility set $C' = \{(m,w) \in C : m \neq a \wedge w \neq b\}$. Are we guaranteed that C' is necessarily plentiful? Briefly justify your answer.
3. Same as in part 2, except now we are told that C is super-plentiful. Prove that, in this case, C' is guaranteed to be plentiful.
4. Prove that if C is plentiful but not super-plentiful, then there exists a non-empty set M of men such that $|f(M)| = |M|$ and $|M| < n$.
5. Suppose that C is plentiful, and moreover that there is a non-empty set M of men such that $|f(M)| = |M|$ and $|M| < n$. Let $k = |M|$. Suppose that those k men (M) and those k women ($f(M)$) hop aboard a space ship to Mars, leaving behind $n-k$ men and $n-k$ women here on Earth. Let $C_{\text{Mars}} = \{(m,w) \in C : m \in M \wedge w \in f(M)\}$ and $C_{\text{Earth}} = \{(m,w) \in C : m \notin M \wedge w \notin f(M)\}$. Prove that C_{Mars} is plentiful.
6. Same as in part 5, but now prove that C_{Earth} is plentiful.

   Hint: If S is a set of male earthlings, what can you say about $|f(S \cup M)|$ vs $|f(S) \setminus f(M)|$?
7. Prove that if C is plentiful, then it has a compatible pairing.

   Hint: Use strong induction and the previous parts.