CS 70 Spring 2008 — Discrete Mathematics for CS — David Wagner — HW 3

Due February 14

The standard instructions apply: you may work in groups of up to four, but you must write your own solutions; your solution must contain your full name, login ID, section number, and list of people you worked with.

1. (6 pts.) A strong induction proof

Let $f(n)$ be defined by the recurrence relation $f(n) = 7f(n-1) - 10f(n-2)$ (for all $n \geq 2$) and $f(0) = 1$, $f(1) = 2$. Prove using strong induction that $f(n) = 2^n$ for every $n \in \mathbb{N}$.

2. (6 pts.) Marbles

In a game that's about to sweep the nation, there is a bucket that contains some number of gold-colored marbles, silver-colored marbles, and bronze-colored marbles. When it is a player's turn, the player may either: (i) remove one gold marble from the bucket, and add up to 3 silver marbles into the bucket; (ii) remove two silver marbles from the bucket, and add up to 7 bronze marbles into the bucket; or, (iii) remove a bronze marble from the bucket. These are the only legal moves. The last player that can make a legal move wins.

Prove by induction that, if the bucket initially contains a finite number of marbles at the start of the game, then the game will end after a finite number of moves.

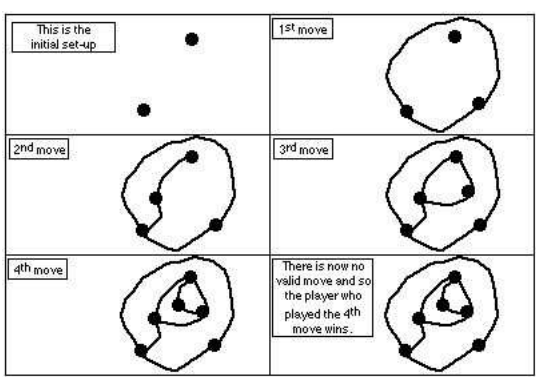
3. (7 pts.) Shall we play a game?

Dots-and-lines is a two-player game played with paper and pencil. Several dots are drawn on the paper. Then the players take turns, each doing the following:

- drawing a line that connects two dots or connects a dot to itself but doesn't touch or cross any other line (without violating the three-line rule below); and,
- putting a new dot on this new line, thus separating it into two lines.

All moves must abide by the following rule: no dot is allowed to have more than three lines attached to it. The last player that can make a legal move wins.

The figure below shows a sample game.



Prove that any game of Dots-and-lines that starts with d dots consists of at most $3d$ moves before someone loses.

Hint: Look for an invariant that is preserved by every legal move.

4. (7 pts.) Let's be social

n people go to a bar. Initially, each person sits at their own table. After a little while, the bartender picks a table, taps the person at that table on the shoulder, and asks him to move to a second table. The person who just moved introduces himself and shakes hands with the person who was already sitting at the second table.

In general, the bartender keeps repeating the following operation: the bartender chooses two tables; the bartender asks everyone sitting at the first table to move over to the second table; and each of the folks who just moved from the first table shake hands with everyone who was already sitting at the second table. Suppose that there were k people sitting at the first table and ℓ people sitting at the second table before this operation. After this operation, there are 0 people at the first table and $k+\ell$ people at the new table. Also, each of the k newcomers shakes hands with each of the ℓ folks already at the second table, so $k\ell$ handshakes occur during this operation. The bartender repeats this kind of operation until all n people are sitting at the same table.

Let $H(n)$ denote the the total number of handshakes that have occurred among the n people by the time this process is finished and everyone is seated at the same table. Prove that it doesn't matter what order the bartender decides to choose tables; we always have $H(n) = n(n-1)/2$.

Hint: Use strong induction.

5. (13 pts.) Reasoning about algorithms

You are given a list $[x_1,\dots,x_n]$ of n integers. The list will be given to you in *sorted order*, so that $x_1 \le x_2 \le \cdots \le x_n$. Also, you are given an integer y. The problem is to find a pair of list elements that sum to y, i.e., to find x_i, x_j such that $x_i + x_j = y$.

Consider the following recursive algorithm for solving this problem:

FindPair($[x_1,\dots,x_n], y$):
1. If $n < 2$, then return NotFound.
2. If $x_1 + x_n = y$ then return (x_1, x_n).
3. If $x_1 + x_n > y$ then return FindPair($[x_1,\dots,x_{n-1}], y$).
4. If $x_1 + x_n < y$ then return FindPair($[x_2,\dots,x_n], y$).

Answer the following questions.

1. List all of the recursive calls to FindPair (and their arguments) that are executed if we call FindPair($[1,6,7,11,13], 18$).
2. If FindPair is executed on a list of $n \in \mathbb{N}$ items, is it guaranteed to terminate? Prove your answer.
3. Prove that if there does not exist any pair of list elements in the list $[x_1,\dots,x_n]$ that sum to y, then FindPair($[x_1,\dots,x_n], y$) returns NotFound.

   Hint: Use induction on the length of the list.
4. Prove that if there exists some pair x_i, x_j of list elements in the list $[x_1,\dots,x_n]$ that sum to y, then FindPair($[x_1,\dots,x_n], y$) returns some pair (x_k, x_ℓ) of list elements such that $x_k + x_\ell = y$.

6. (7 pts.) Who wants to be a millionaire?

Bill Gates arranges n^2 gold coins in a $n \times n$ grid, each with their "heads" side up. He invites you to play a game: in each move, you can pick one of the n rows or columns and flip all of the gold pieces in that row or column. You are not allowed to re-arrange the coins in any other way. If you can reach a configuration where there is exactly one gold coin with its "tails" side up, Bill promises to award you as a prize all n^2 of the gold pieces. He generously allows you an unlimited number of moves. For which values of n is it possible to win the game? Prove your answer.

7. (4 pts.) You be the grader

Assign a grade of A (correct) or F (failure) to the following proof. If you give a F, please explain clearly where the logical error in the proof lies. Saying that the claim is false is *not* a valid explanation of what is wrong with the proof. If you give an A, you do not need to explain your grade.

1. **Claim:** For every $n \in \mathbb{N}$, $2\cdot 3+3\cdot 4+4\cdot 5+\cdots+n\cdot(n+1) = n(n+1)(n+2)/3$.

   Proof: The proof will be by induction on n. Let $P(n)$ denote the proposition that $2\cdot 3+3\cdot 4+\cdots+n\cdot(n+1) = n(n+1)(n+2)/3$.
   Base case: $P(0)$ is true, since both sides of the equation are equal to zero.
   Induction hypothesis: Suppose $P(n)$ is true, i.e., $2\cdot 3+3\cdot 4+\cdots+n\cdot(n+1) = n(n+1)(n+2)/3$.
   Induction step: We must prove $P(n+1)$. We can calculate

$$\begin{aligned} 2\cdot 3+3\cdot 4+\cdots+n\cdot(n+1)+(n+1)\cdot(n+2) &= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) \\ &= (n+1)(n+2)\left(\frac{n}{3}+1\right) \\ &= \frac{(n+1)(n+2)(n+3)}{3}. \end{aligned}$$

   (using the induction hypothesis on the first line), so $P(n+1)$ follows from $P(n)$. Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n. $\square$