CS 70 Spring 2008 | Discrete Mathematics for CS David Wagner | **HW 2**

Due February 7

As a reminder, your submission should start with the following information:

Your full name
Your login name
The name of the homework assignment (e.g. hw2)
Your section number
The list of people you worked with on this homework, or "none" if you didn't work with anyone

As for all homework assignments in CS70, you are welcome to work on the homework in small groups (up to four people), but you **must** write up all your solutions on your own.

1. (16 pts.) A few proofs

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in the Lecture Notes) you used.

1. For all natural numbers n, if n is even then n^5 is even.
2. For all natural numbers n, $n^2 - n + 3$ is odd.
3. For all real numbers x, y, if $\frac{x+y}{2} \geq 10$ then $x \geq 10$ or $y \geq 10$.
4. For all real numbers r, if r is irrational then r^2 is irrational.

2. (5 pts.) Proof by induction

For $n \in \mathbb{N}$ with $n \geq 2$, define s_n by

$$s_n = (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{n}).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

3. (5 pts.) Another induction proof

Let $a_n = 3^{n+2} + 4^{2n+1}$. Prove that 13 divides a_n for every $n \in \mathbb{N}$.

(Hint: What can you say about $a_{n+1} - 3a_n$?)

4. (6 pts.) Tower of Brahma

This puzzle was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

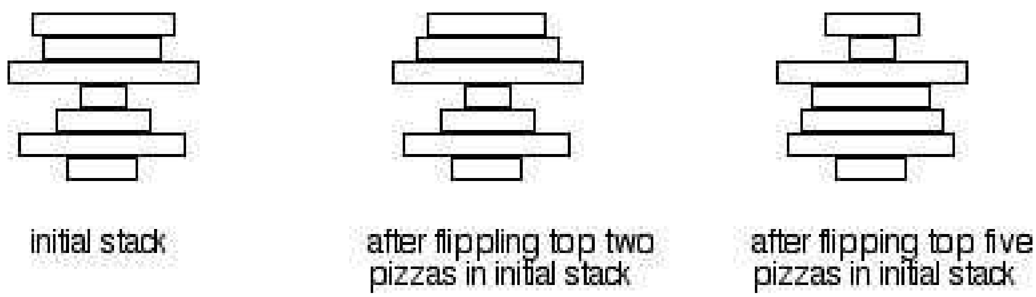
In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and

Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

Prove by induction the exact number of moves required to carry out this task in general, if there are n disks on the original needle. Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy predict before the destruction of the World?

5. (6 pts.) The proof of the pi is in the eating

Over the summer, Dave Wagner takes a job at the local pizza parlor, where he tends to be a bit distractable. One day, he has a stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size, with the largest pizza on the bottom, the next largest pizza just above that, and so on. He has learned how to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). The figure below shows two sample flips.



This is the only move Dave can do to change the order of the stack; however, he is willing to keep repeating this move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some sequence of moves, no matter how many pizzas he starts with? Prove your answer.

6. (12 pts.) You be the grader

Assign a grade of A (correct) or F (failure) to the following proofs. If you give a F, please explain clearly where the logical error in the proof lies. Saying that the claim is false is *not* a valid explanation of what is wrong with the proof. If you give an A, you do not need to explain your grade.

1. **Claim:** For every $n \in \mathbb{N}$, $n^2 + 3n$ is odd.
   Proof: The proof will be by induction on n.
   Base case: The number $n = 1$ is odd.
   Induction step: Suppose $k \in \mathbb{N}$ and $k^2 + 3k$ is odd. Then,

   $$(k+1)^2 + 3(k+1) = (k^2 + 2k + 1) + (3k + 3) = (k^2 + 3k) + (2k + 4)$$

   is the sum of an odd and an even integer. Therefore, $(k+1)^2 + 3(k+1)$ is odd. Therefore, by the principle of mathematical induction, $n^2 + 3n$ is odd for all natural numbers n. □

2. **Claim:** For every real number x, if x is irrational, then $2008x$ is irrational.
   Proof: Suppose $2008x$ is rational. Then $2008x = p/q$ for some integers p, q with $q \neq 0$. Therefore $x = p/(2008q)$ where p and $2008q$ are integers with $2008q \neq 0$, so x is rational. Therefore, if $2008x$ is rational, then x is rational. By the contrapositive, if x is irrational, then $2008x$ is irrational. □

3. **Claim:** For every $n \in \mathbb{N}$, if $n \geq 4$, then $2^n < n!$.

   Proof: The proof will be by induction on n.
   Base case: $2^4 = 16$ and $4! = 24$ and $16 < 24$, so the statement is true for $n = 4$.
   Induction step: Suppose $k \in \mathbb{N}$ and $2^k < k!$. Then

   $$2^{k+1} = 2 \times 2^k < 2 \times k! < (k+1) \times k! = (k+1)!,$$

   so $2^{k+1} < (k+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$. □

4. **Claim:** For all $x, y, n \in \mathbb{N}$, if $\max(x,y) = n$, then $x \leq y$.

   Proof: The proof will be by induction on n.
   Base case: Suppose that $n = 0$. If $\max(x,y) = 0$ and $x, y \in \mathbb{N}$, then $x = 0$ and $y = 0$, hence $x \leq y$.
   Inductive hypothesis: Assume that, whenever we have $\max(x,y) = k$, then $x \leq y$ must follow.
   Inductive step: We must prove that if $\max(x,y) = k+1$, then $x \leq y$. Suppose x, y are such that $\max(x,y) = k+1$. Then it follows that $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 \leq y - 1$. In this case, we have $x \leq y$, completing the induction step. □