CS 70	Discrete Mathematics for CS	
Spring 2008	David Wagner	HW 11

Due Thursday, April 24th

Important: Show your work on all problems on this homework.

1. (5 pts.) Random permutations

Write down the numbers $1,2,\ldots,n$ in a random order, with all $n!$ orders equally likely. Underline each number in the sequence that is smaller than the number immediately following it. Let the random variable X count the number of underlined numbers. For instance, if $n = 5$, one possible sequence is $5,3,2,4,1$, and after underlining we get $5,3,\underline{2},4,1$, so in this example $X = 1$.

Compute $\mathbf{E}[X]$. Your answer should be expressed as a simple function of n.

2. (10 pts.) A LCD display

We have a LCD display with 8×40 pixels, i.e., 8 rows with 40 pixels in each row. The display has a vertical line in some column if all 8 pixels in a particular column are turned on. Suppose that each pixel is turned on or off with equal probability, and all pixels are independent. Let the r.v. X denote the number of vertical lines in the display.

(a) Calculate $\mathbf{E}[X]$.

(b) Calculate $\mathrm{Var}(X)$.

3. (15 pts.) Chopping up DNA

(a) In a certain biological experiment, a piece of DNA consisting of a linear sequence (or string) of 4000 nucleotides is subjected to bombardment by various enzymes. The effect of the bombardment is to randomly cut the string between pairs of adjacent nucleotides: each of the 3999 possible cuts occurs independently and with probability $\frac{1}{500}$. What is the expected number of pieces into which the string is cut? Justify your calculation.

[Hint: Use linearity of expectation! If you do it this way, you can avoid a huge amount of messy calculation. Remember to justify the steps in your argument; i.e., do not appeal to "common sense."]

(b) Suppose that the cuts are no longer independent, but highly correlated, so that when a cut occurs in a particular place other cuts close by are much more likely. The probability of each individual cut remains $\frac{1}{500}$. Does the expected number of pieces increase, decrease, or stay the same? Justify your answer with a precise explanation.

(c) Let the r.v. $X =$ the number of pieces into which the string is cut in part (a), where all cuts occur independently. Calculate $\mathrm{Var}(X)$.

4. (15 pts.) The martingale

Consider a *fair game* in a casino: on each play, you may stake any amount \$$S$; you win or lose with probability $\frac{1}{2}$ each (all plays being independent); if you win you get your stake back plus \$$S$; if you lose you lose your stake.

(a) What is the expected number of plays before your first win (including the play on which you win)?

(b) The following gambling strategy, known as the "martingale," was popular in European casinos in the 18th century: on the first play, stake \$1; on the second play \$2; on the third play \$4; on the kth play $\$2^{k-1}$. Stop (and leave the casino!) when you first win.

Show that, if you follow the martingale strategy, and assuming you have unlimited funds available, you will leave the casino \$1 richer with probability 1. [Maybe this is why the strategy is banned in most modern casinos.]

(c) To discover the catch in this seemingly infallible strategy, let X be the random variable that measures your maximum loss before winning (i.e., the amount of money you have lost *before* the play on which you win). Show that $\mathbf{E}[X] = \infty$. What does this imply about your ability to play the martingale strategy in practice?

5. (5 pts.) Grade this proof

What's wrong with this proof? Please explain the flaw in this writeup.

Theorem: For every $n \in \mathbb{N}$ with $n \geq 2$, we have $\sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}$.

Proof: We will use (simple) induction on n.
Base case: If $n = 2$, then $\sum_{i=2}^{2} \binom{i}{2} = \binom{2}{2} = 1 = \binom{3}{3}$.
Inductive hypothesis: Suppose $\sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}$ for every $n \in \mathbb{N}$ with $n \geq 2$.
Inductive step: We must show that $\sum_{i=2}^{n+1} \binom{i}{2} = \binom{n+2}{3}$. Let's calculate:

$$\begin{aligned}
\sum_{i=2}^{n+1} \binom{i}{2} &= \left(\sum_{i=2}^{n} \binom{i}{2} \right) + \binom{n+1}{2} \\
&= \binom{n+1}{3} + \binom{n+1}{2} \\
&= \binom{n+2}{3}.
\end{aligned}$$

In the second line, we used the inductive hypothesis. In the third line, we used the fact that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$, which follows from this calculation:

$$\begin{aligned}
\binom{a}{b} + \binom{a}{b+1} &= \frac{a!}{b!(a-b)!} + \frac{a!}{(b+1)!(a-b-1)!} \\
&= \frac{a! \cdot (b+1)}{(b+1)!(a-b)!} + \frac{a! \cdot (a-b)}{(b+1)!(a-b)!} \\
&= \frac{a! \cdot (a+1)}{(b+1)!(a-b)!} \\
&= \frac{(a+1)!}{(b+1)!(a-b)!} \\
&= \binom{a+1}{b+1}.
\end{aligned}$$

This concludes the proof. □

6. (0+5 pts.) Optional bonus puzzle

We're going to play a game. You have a team of 7 people, seated around a circular table. When the game begins, the referee is going to put a colored hat on each person's head, with each color chosen from a palette of 7 alternatives (so there are 7^7 combinations of hats possible). Each player can see everyone else's hats, but no one can see the color of their own hats. (There are no mirrors.) Also, once the game begins, the players are not allowed to communicate by any means whatsoever (no hand signals, no footsie, nothing). However, you may coordinate before the game begins on the strategy that you will use. After everyone receives their hats, each player is handed a slip of paper, and after looking at everyone else's hat colors, the player secretly writes down a guess at his own hat color.

Once everyone has marked their slip of paper, the referee checks them to see whether the team wins. If everyone guessed incorrectly, the whole team loses. If at least one person guessed correctly, the whole team wins. Note that you either win or lose as a team, i.e., either everyone wins or everyone loses. Therefore it is in everyone's interests to cooperate.

Question: Identify a strategy that ensures a *guaranteed* win, i.e., so that the team is certain to win no matter which of the 7^7 combinations of hat colors the referee picks.