CS 70 Spring 2008 — Discrete Mathematics for CS — David Wagner — HW 10

Due Thursday, April 17th

Important: Show your work on all problems on this homework.

1. (10 pts.) A paradox in conditional probability?

Here is some on-time arrival data for two airlines, A and B, into the airports of Los Angeles and Chicago. (Predictably, both airlines perform better in LA, which is subject to less flight congestion and less bad weather.)

	Airline A		Airline B	
	# flights	# on time	#flights	# on time
Los Angeles	600	534	200	188
Chicago	250	176	900	685

(a) Which of the two airlines has a better chance of arriving on time into Los Angeles?

(b) Which of the two airlines has a better chance of arriving on time into Chicago?

(c) Which of the two airlines has a better chance of arriving on time overall?

(d) Explain the apparent paradox intuitively.

(e) Interpret this in terms of conditional probabilities. If $\Pr[A|E] > \Pr[B|E]$ and $\Pr[A|\overline{E}] > \Pr[B|\overline{E}]$, are we guaranteed that $\Pr[A] > \Pr[B]$?

2. (5 pts.) Games

Here's a game. Alice and Bob will each roll a fair, six-sided die. If Alice's die comes up with a number higher than Bob's, Alice wins \$3 from Bob. If Bob's number comes up higher, or if they tie, Bob wins \$2 from Alice. Is this game a good deal for Alice? Explain.

(Hint: Compute an expected value.)

3. (20 pts.) Mean time to failure

I have a light bulb. Suppose that it has a probability p of burning out on any particular day, if it has not burnt out already. Let the random variable D represent the number of days until it burns out, so that $\Pr[D=0] = p$, $\Pr[D=1] = p(1-p)$, and so on.

(a) If n is a positive integer, what is $\Pr[D=n]$ (as a simple function of n and p)?

(b) If n is a non-negative integer, what is $\Pr[D>n]$ (as a simple function of n and p)?

(c) Prove that if X is any random variable that takes values on $\mathbb{N}$, then $\mathbf{E}[X] = \sum_{n=0}^{\infty} \Pr[X>n]$.

(d) Calculate the expected time until the light bulb burns out, namely, $\mathbf{E}[D]$ (as a simple function of p).

(e) A couple desperately wants a baby boy, so they decide to keep having children until they have a boy. Each child they have has a 50% chance to be a boy (independent of the gender of all prior children). What's the expected number of baby girls they will have, before they have their first baby boy?

4. (15 pts.) Another game

Here's a fun game for three players, Alice, Bob, and Charles. First, each player puts \$2 on the table and secretly writes down either Heads or Tails on a slip of paper (without allowing the other players to see it). Then someone tosses a fair coin. Finally, the \$6 on the table is split evenly among everyone who correctly predicted the result of the coin toss. If everyone guessed wrong, everyone takes back their \$2.

(a) Assume that each of the three players randomly and independently decides what to write on his/her slip of paper by writing Heads or Tails with equal probability. Let the random variable X represent the profit that Charles makes (in dollars) when playing one round of this game, i.e., $X =$ the number of dollars that Charles wins minus \$2. Find the sample space Ω and identify the value of X for each sample point $\omega \in \Omega$.

(b) Calculate $\mathbf{E}[X]$. Is the game fair to Charles? Explain.

(c) Next, let's consider what happens if Alice and Bob collude in the following sneaky way. They'll prearrange their strategy to ensure that Bob's guess is always the opposite of Alice's guess. Let the random variable Y represent Charles' profit when he plays one round of the game in this scenario. Calculate $\mathbf{E}[Y]$. Is this fair to Charles? Explain.

(d) A small group of your friends are organizing a hockey pool, where each day you all bet on the outcome of a hockey playoff game relative to the spread. Whoever correctly predicts the outcome of the most number of games, wins the pot. Assume that bookkeepers are so good at setting the spread that you are each effectively betting on the outcome of a fair coin toss each day.

Suppose your friends haven't taken CS 70 yet, so they're unaware of the dirty tricks we teach in this class. Based on the analysis above, suggest a way that you could gain an unfair advantage so you will win the pot with probability $> 1/n$, where n is the number of participants in the pool. You do not need to formally prove that your method will work or calculate exactly what your chance of winning is, but please give some informal explanation why it gives you an unfair advantage.