The role of initial conditions in the synchronization problem

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Abstract

We show the numerical evidence of a case of desynchronized behaviour in two mutual y-coupled Chua's circuits with negative conditional Lyapunov exponents (CLE). In this case the synchronized state strongly depends on the initial conditions in the two circuits according to the shape of the basin of attraction. Crossing the basin's boundaries from synchronized to desynchronized phase, the maximum CLE, even showing a sharp transition, has always a negative value. We suggest two different ways to calculate the CLE to obtain more information about the desynchronized state.

Synchronization of chaotic systems had received a lot of attention in the last few years, for the case of drive-response systems, since the first pioneering works of Fujisaka and Yamada, and Pecora and Carroll (PC) [1, 2]. This property has been analyzed both from the theoretical and experimental point of view, specially regarding simulations of numerical models associated to different experimental situations observed in real physical chaotic systems [3, 4, 5, 6]. In the early works, PC assumed that the synchronization of two *identical* response systems under a common drive signal can be numerically predicted from the sign of the *conditional Lyapunov exponents* (CLE), defined as the exponents of the response system in terms of their own variables only. Great efforts in this field had produced the extension of this idea to response systems radically different [7]. The aim of present work is to discuss a case of desynchronization occurring when two *mutual* coupled chaotic systems display negative CLE, proposing a new way to obtain information on the desynchronization by the study of the variation of the CLE. Since the first work of Kowalski et al. [8], it seems plausible that the PC's ideas can be extended to this class of systems. We believe that the synchronization condition for drive-response systems works well even for mutual coupled chaotic systems. We study the dynamical behaviour of two coupled Chua's circuits [9] focusing our attention on the basin of attraction, namely the set of initial conditions whose evolution converges to the same attractor. Their knowledge will permit the definition of *complete* topological invariants as fractal dimension and Lyapunov exponents. The choice of the relatively simple Chua's model is due to the possibility to compare our theoretical results with experimentally controlled outputs [10]. The rescaled equations of motion of the two identical y-coupled Chua's circuits are:

$$\begin{cases} \dot{x} = \alpha(y - x - f(x)) \\ \dot{y} = x - y + z - \delta(y - y') \\ \dot{z} = -\beta y - \gamma z \\ \dot{x}' = \alpha(y' - x' - f(x')) \\ \dot{y}' = x' - y' + z' + \delta(y - y') \\ \dot{z}' = -\beta y' - \gamma z' \end{cases} \tag{1}$$

where $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$ is the transformed functional representing the $v - i$ characteristic of the Chua's diode and δ is the coupling constant. All quantities in Eqs. (1) are dimensionless. We study the case of two identical dynamical systems mutually

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coupled with a matrix $A \in \mathcal{R}^{n\times n}$, so that

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \mathbf{f}(\mathbf{r}) - A(\mathbf{r}-\mathbf{r}') \\ \frac{d\mathbf{r}'}{dt} &= \mathbf{f}(\mathbf{r}') + A(\mathbf{r}-\mathbf{r}')\end{aligned} \quad (2)$$

where $\mathbf{r}, \mathbf{r}' \in \mathcal{R}^n$ are the variables and $\mathbf{f}$ is the functional form of the two systems. We define $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}'$ so that the equations (2) can be expressed also in term of the difference of the corresponding variables. Let δ be the only variable parameter in the matrix A, and $\mathbf{R} \in \mathcal{R}^{2n}$ be the vector $\mathbf{r} \otimes \Delta\mathbf{r}$; with this definition the equations (2) can be brought in the form

$$\frac{d\mathbf{R}}{dt} = \mathbf{F}(\mathbf{R}, \delta). \quad (3)$$

The functional form $\mathbf{F}$, when the coupling is off ($A_{ij} = 0, \forall i, j = 1, n$), can be seen like the carthesian product $\mathbf{F}(\mathbf{R}) = \mathbf{f}(\mathbf{r}) \otimes \mathbf{g}(\mathbf{r}, \Delta\mathbf{r})$, where $\mathbf{g}$ is the functional form of $\dot{\Delta\mathbf{r}} = \mathbf{g}(\mathbf{r}, \Delta\mathbf{r})$, namely the differential equations for the *difference* variables. For calculating the Lyapunov exponents from the equations (3) we need to integrate the corresponding variational equations,

$$\frac{d\xi}{dt} = DF(\mathbf{R}, \delta)\xi. \quad (4)$$

The matrix $DF(\mathbf{R}, \delta) \in \mathcal{R}^{2n\times 2n}$, can be viewed as composed of several blocks:

$$DF(\mathbf{R}, \delta) = \begin{pmatrix} \nabla_{\mathbf{r}}\mathbf{f} & -A \\ \nabla_{\mathbf{r}}\mathbf{g} & \nabla_{\Delta\mathbf{r}}\mathbf{g} - 2A \end{pmatrix} \quad (5)$$

The upper left block is the Jacobian matrix of the first three independent equations, while on the right is the coupling matrix A. Down on the left there is the matrix composed of derivatives of the difference equations in terms of the $\mathbf{r}$ variables. For algebric nonlinearities up to the third order, this block is composed by only difference variables and in the case of identical synchronized behaviour all terms go to zero. The last block is the Jacobian matrix of the difference equations in terms of their own variables only. From now on, we calculate the CLE in two different ways:

1. Following the ideas of PC who calculated the characteristic Lyapunov exponents of the differences equations in the case of drive-response systems, we used in our evaluation of the CLE only the lower right block $\nabla_{\Delta\mathbf{r}}\mathbf{g} - 2A$ of the matrix (5). This point of view is an approximation, due to the fact that the contribution of the terms in the lower left block of the matrix (5) has been neglected. In our case this block $\nabla_{\mathbf{r}}\mathbf{g}$ has only one nonlinear term different from zero, exactly the term

$$(\nabla_{\mathbf{r}}\mathbf{g})_{11} = DF(\mathbf{R}, \delta) = -\alpha\frac{\partial}{\partial x}[f(x) - f(x')].$$

The approximation works well in the case of identical synchronized behaviour, for which the neglected terms give no contribution. Working in this way we analyze the stability of the attractor in the invariant subset by computing the evolution of the perturbations transverse to the subset $\Delta\mathbf{r} = 0$ [11]. 2. To evaluate the quantitative validity of the previous approximation, we have calculated the CLE also by including the $\nabla_{\mathbf{r}}\mathbf{g}$ term. This can be done integrating the Eqs. (4), applying the Gram-Schmidt procedure, but orthogonalizing only the $\Delta\mathbf{r}$ variables. We think that this method gives a better approximation for the results than the method 1. The results will be discussed later on. The numerical simulations have been carried out by integrating the Eqs. (1) whose coupling corresponds to a matrix A in Eqs.(2) having the only $A_{22} = \delta$ element different from zero (y-coupling). To investigate numerically the Eqs.(1) we pose $\alpha = 9.0$, $\beta = 14.8$, $\gamma = 0.015$, $a = -1.14$ and $b = -0.72$ for different values of δ . As a general result we find that the choice of the initial conditions strongly affects the property of synchronization. For this reason, we fix the initial conditions for the first three equations by putting $\mathbf{r}_0 = (0.1, 0.1, 0.0)$, and use the simple rule

$$\begin{aligned}x'_0 &= x_0 + \Delta x_0 \\ y'_0 &= y_0 + \Delta y_0 \\ z'_0 &= z_0.\end{aligned} \quad (6)$$

The resulting basin of attraction for $\delta = 3.0$, is shown in Fig. 1. Changing the value of δ in the interval $[2; 5]$, the pictures of the basin are closely similar to

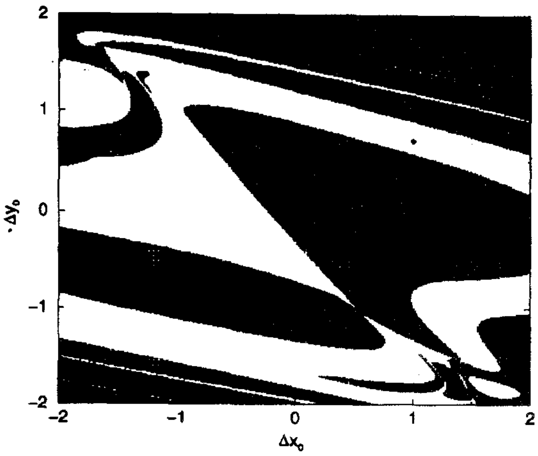

Figure 1: Basin of attraction for the coupled Chua's circuits in the $(\Delta x_0, \Delta y_0)$ plane. The synchronous regions are in black, while the desynchronized ones are in blank. The initial conditions in the first circuit are $x_0 = 0.1, y_0 = 0.1$ and $z_0 = 0.0$, the coupling parameter is δ=3.0. The two dots represent the values of initial conditions for the attractors in Fig. 2

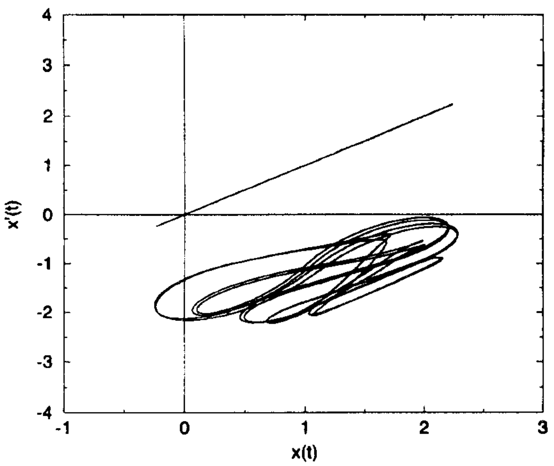

Figure 2: Upper curve: synchronized behaviour at $\Delta y_0 = 0.5$ in Fig. 1. Lower curve: periodic behaviour of the desynchronized state at $\Delta y_0 = 0.7$. In both cases $\Delta x_0 = 1.0$.

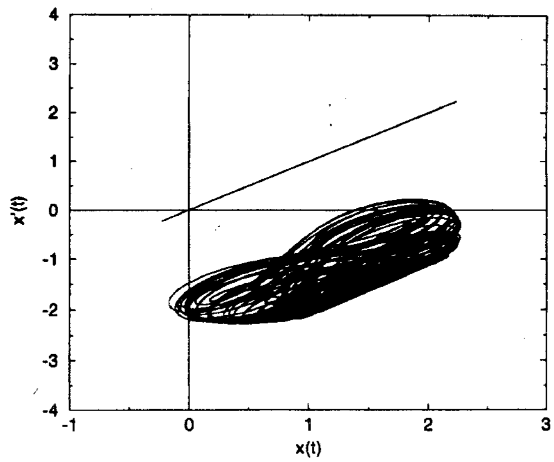

Figure 3: Same as in Fig. 2 but with $\delta = 4.0$. In this case the desynchronized state is chaotic.

this without any evident topological variation. The black regions mark zones of perfect identical synchronization between the variables of the two coupled equations, while the white one defines the set of $(\Delta x_0, \Delta y_0)$ for which the Eqs.(1) do not synchronize. In the Fig. 2 and Fig. 3 we show the x and x' compositions obtained by integrating the Eqs.(1) in two different regions of the basins for different choices of the coupling parameter. When the systems are synchronized the two dynamics converge to the same manifold and the xx' compositions give a straight line. In all cases $\Delta x_0 = 1.0$, and in the figures are shown the Δy_0 values that correspond to different states. It's also evident that the desynchronization for $\delta = 3.0$ is periodic. Fig. 4 shows the value of the maximum CLE for $\Delta x_0 = 1.0$ and with Δy_0 crossing twice the basin boundaries. In all cases the result is that the value of the maximum CLE is always negative, in spite of the radically different dynamical behaviour. The remarkable result shown in Fig. 4 is that the value of maximum CLE calculated with and without the off-diagonal term gives the same result in the synchronization region $\mathbf{\Delta r}$, as expected, and remains in any case negative by changing numerically its absolute value in the desynchronized region.

For $\delta = 3.0$, in the borders of the desynchronization interval, the CLE oscillate between two values: the xx' compositions of the two dynamics show two equal but symmetric attractors. Moreover, varying the parameters we find a basin of attraction with *riddled* shape (Fig. 5) i.e. the sets of initial condition attracted to different dynamics are dense one in the other [12]. Also in this case the maximum CLE and the transverse exponents are negative [13]. Recently [14], it has been

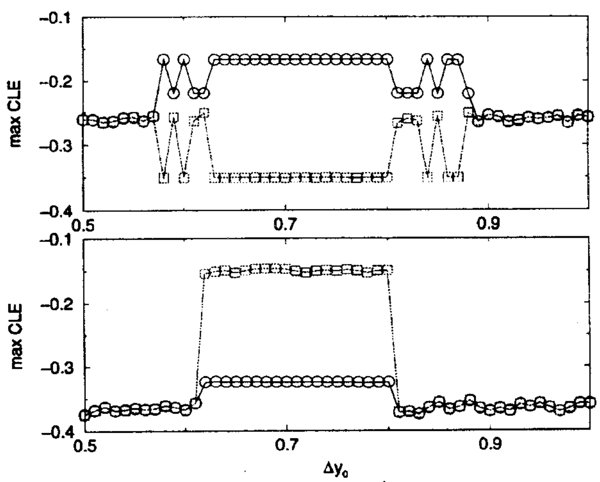

Figure 4: Maximum CLE. The always negative value corresponds to $\delta = 3.0$ (up), $\delta = 4.0$ (down) for Δy_0 crossing the basin boundaries at $\Delta x_0 = 1.0$. In all figures the circles (squares) correspond to the results obtained without (with) the off-diagonal $\nabla_{\mathbf{r}}\mathbf{g}$ term.

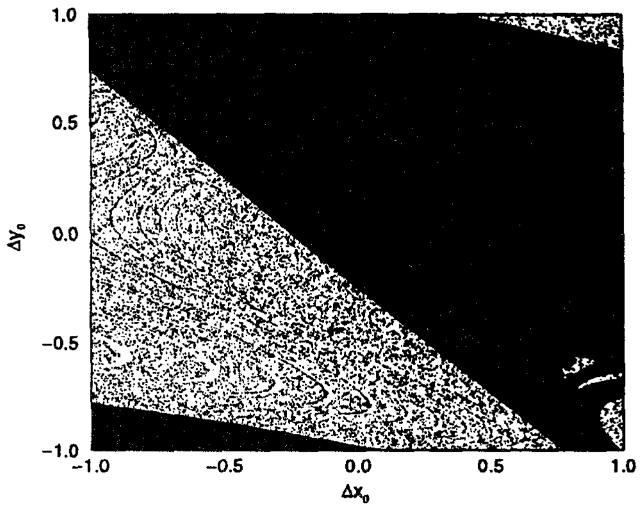

Figure 5: Basin of attraction as in Fig. 1 showing the evidence of riddling for the synchronization. The parameter values for this case are $\alpha = 10$, $\beta = 15.125$, $\gamma = 0.2248\bar{3}$, $a = -1.2507$, $b = -0.67845$ and $\delta = 1.65$.

conjectured that the negativity of the maximum CLE, in the case of drive-response systems and unidirectionally coupled systems, could be related to the presence not only of identical synchronization (IS), but also of GS. Two systems show GS if their trajectories asymptotically converge not in the manifold of identical corresponding variables, $\mathbf{r}=\mathbf{r}'$, but in a functional relation between the two set of variables, $\mathbf{r}=\phi(\mathbf{r})$. The result, in our case of mutually coupled systems, is that the presence of the GS cannot be predicted by the only knowledge of the CLE.

In this paper we have investigated the dynamical behaviour of the Eqs.(1), representing two mutual y-coupled Chua's circuits, looking for the dependence of the synchronized states from the choice of the initial conditions, and its connections with the CLE. We extend the PC's assumption to non drive-response systems. Also for mutually coupled systems the negativity of CLE is only a necessary but not sufficient condition for the synchronization. The strong dependence of the synchronization condition from the set of initial condition forces us to study in great details the attractions basins of the system that in some cases may have a riddled shape. This fact, as shown in Fig.5, constitutes a further difficulty source for a comprehensive study of this very intriguing problem.

References

[1] H. Fujisaka and T. Yamada, *Prog. Theor. Phys.* 69, 32 (1983).

[2] L.M. Pecora and T.L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990); *Phys. Rev. A* **44**, 2374 (1991).

[3] H.G. Winful and L. Rahman, *Phys. Rev. Lett.* **65**, 1575, (1990); R. Roy and K.S. Thornburg, *Jr. Phys. Lett.* **72**, 2009 (194).

[4] K. Cuomo and A.V. Oppenehim, *Phys. Rev. Lett.* **71**, 65 (1993); N.F. Rul'kov, A.R. Volkovskii A. Rodriguez -Lozano, E. d. Rio and M.G. Velarde, *Int. J. Bif. and Chaos* **2**, 669 (1992).

[5] H. Sompolinsky, D. Golomb and D. Kleinfeld, *Phys.Rev. A* **43**, 6990 (1991).

[6] G.B. Ermentrout, *Physica D* **41**, 219 (1990).

[7] M. Ding and E. Ott, *Phys. Rev. E* **49**, 945 (1994).

[8] J.M. Kowalski, G.L. Albert and G.W. Gross, *Phys. Rev. A* **42**, 6260 (1990).

[9] For a complete review see *Chua's circuit: a paradigm for chaos*, edited by R.N. Madan, (World Scientific, Singapore, 1993).

[10] A. Varone, G. Santoboni, G. Pegna and F. Meloni, to be published.

[11] T. Kapitaniak and K.E. Thylwe, *Chaos, Solitons & Fractals* **10**, 1569 (1996).

[12] J.C. Alexander, J.A. Yorke, Z. You and I. Kan, sl Int. J. of Bifur. and Chaos **2**, 795 (1992); R.H. Parmenter and L.Y. Yu, *Phys. Lett A* **189**, 181 (1994); J.F. Heagy, T.L. Carroll and L.M. Pecora, *Phys. Rev. Lett.* **73**, 3528 (1994).

[13] A. Varone, G. Santoboni and F. Meloni, unpublished.

[14] L. Kocarev and U. Parlitz, *Phys. Rev. Lett.* **74**, 5028 (1995); *Phys. Rev. Lett.* **76**, 1816 (1996) ; N.F. Rulkov M.M. Sushchik, L.S. Tsmiring and H.D.I. Abarbanel, *Phys. Rev. E* **51**, 980 (1995).