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MASTER-SLAVE SYNCHRONIZATION OF LUR'E SYSTEMS

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In this paper we present a sufficient condition for master-slave synchronization of Lur'e systems. The scheme makes use of linear full static state feedback. The criterion is based on a Lur'e–Postnikov Lyapunov function for global asymptotic stability of the error system. The condition is basically the same as the one for global asymptotic stability of the Lur'e system, controlled with linear state feedback. The design of the feedback matrix is done by solving a constrained nonlinear optimization problem. The method is illustrated on the synchronization of Chua's circuit.

1. Introduction

Recently there is a lot of interest in the use of synchronization for secure communication applications. In [Hasler, 1994] an overview of methods for synchronization is presented, including decomposition into subsystems, linear feedback and an inverse system approach. In order to use synchronization in transmission systems, a useful information-carrying signal is transmitted, hidden in a chaotic signal. Methods for hiding the information are, e.g. chaotic masking, chaotic switching and direct chaotic modulation.

The work described in this paper is related to synchronization by linear feedback, which investigates the problem from the viewpoint of control theory. It has already been shown that there is a close relationship between the synchronization problem and the problem of controlling chaos by linear feedback [Chen, 1993; Hasler, 1994; Wu & Chua, 1994]. The same observation will be made here based upon a matrix inequality, that expresses a sufficient condition for global asymptotic stability of the error system. Conditions for synchronization of general nonlinear systems have been derived for quadratic Lyapunov functions [Wu & Chua, 1994, 1995]. The condition proved in this paper, is derived from a Lur'e–Postnikov Lyapunov function, which consists of a quadratic Lyapunov function plus integral term. We consider Lur'e systems which consist of a linear dynamical system, feedback interconnected to a static nonlinearity that satisfies a sector condition. Many nonlinear systems can be represented in this form, including e.g. Chua's circuit [Guzelis & Chua, 1993]. The fact that the nonlinearities satisfy a sector condition is exploited to derive the matrix inequality. It also yields a systematic procedure for designing the feedback matrix, by solving a constrained nonlinear optimization problem. Finally,

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in order to use this method for chaotic masking, Lur'e systems that reveal high dimensional chaos should be used in order to obtain secure communication schemes. Generalized cellular neural networks that generate high dimensional chaos are possible candidates for that purpose: they have been represented as Lur'e systems in [Guzelis & Chua, 1993].

This paper is organized as follows. In Sec. 2 we present the matrix inequality condition for synchronization of Lur'e systems. In Sec. 3 we illustrate the method on synchronizing Chua's circuit.

2. Matrix Inequality for Synchronization of Lur'e Systems

Let us consider a Lur'e system, which is of the form [Boyd *et al.*, 1994; Khalil, 1992; Narendra & Taylor, 1973]:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= \sigma(y)\,, \end{aligned} \tag{1}$$

consisting of a linear dynamical system, feedback interconnected to static nonlinearities $\sigma_i(\cdot)$ that satisfy sector condition $[0,\, k]$ for all $i = 1, \ldots, n_h$; with state vector $x(t) \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_h}$, $C \in \mathbb{R}^{n_h\times n}$. Using two identical Lur'e systems in a master-slave synchronization scheme with linear full static state feedback, one has:

$$\begin{cases} (\mathcal{M})\, \dot{x} = Ax + B\sigma(Cx) \\ (\mathcal{S})\, \dot{z} = Az + B\sigma(Cz) + F(x - z) \end{cases} \tag{2}$$

with master $\mathcal{M}$ and slave $\mathcal{S}$ and feedback matrix $F \in \mathbb{R}^{n\times n}$. The aim of synchronization is then to obtain $\|x(t) - z(t)\| \to 0$ for time $t \to \infty$. Defining the error signal $e = x - z$, one obtains the error system:

$$\dot{e} = (A - F)e + B\eta(Ce) \tag{3}$$

with $\eta(Ce) = \sigma(Ce + Cz) - \sigma(Cz)$. In the sequel we use the shorthand notation η for $\eta(Ce)$. Assume a sector condition $[0,\, k]$ on $\eta(\cdot)$ which gives the following inequalities for η:

$$\eta_i[\eta_i - kc_i^T e] \le 0, \quad i = 1, \ldots, n_h \quad \forall x,\, z \in \mathbb{R}^n\,, \tag{4}$$

where c_i^T denotes the ith row of C.

Now we investigate under what condition the error signal goes to zero, whatever the choice of the initial states $x(0)$, $z(0)$. Therefore we take the Lur'e–Postnikov Lyapunov function:

$$V(e) = e^T Pe + \sum_{i=1}^{n_h} 2\gamma_i \int_0^{c_i^T e} \eta_i(\rho) k d\rho \tag{5}$$

with $P = P^T > 0$ and $\gamma_i \ge 0$ (assume z is quasi constant with respect to error dynamics). This function is positive everywhere and radially unbounded and is used in order to show under which condition the error system is globally asymptotically stable with unique equilibrium point $e = 0$. The following Theorem is obtained:

Theorem. *Let* $\Gamma = \mathrm{diag}\{\gamma_i\}$, $T = \mathrm{diag}\{\tau_i\}$ *be diagonal matrices with* $\gamma_i, \tau_i \ge 0$ *for* $i = 1, \ldots, n_h$. *Then, if there exist* $P = P^T > 0$, Γ, T *and a feedback matrix* F *such that*

$$Y = Y^T = \begin{bmatrix} (A-F)^T P + P(A-F) & PB + kC^T T + k(A-F)^T C^T \Gamma \\ B^T P + kTC + k\Gamma C(A-F) & -2T + k\Gamma CB + kB^T C^T \Gamma \end{bmatrix} < 0 \tag{6}$$

then based upon (5) the system (2) synchronizes, with error system (3) having a unique and globally asymptotically stable equilibrium point $e = 0$.

Proof. Taking the time derivative of (5) and applying the $\mathcal{S}$-procedure [Boyd *et al.*, 1994] by using the inequalities (4), one obtains:

$$\begin{aligned} \dot{V} = \dot{e}^T Pe + e^T P\dot{e} + \sum_i 2\gamma_i\eta_i(c_i^T e)kc_i^T \dot{e} &\le [(A-F)e + B\eta]^T Pe + e^T P[(A-F)e + B\eta] \\ &+ \sum_i 2\gamma_i\eta_i(c_i^T e)kc_i^T[(A-F)e + B\eta] - \sum_i 2\tau_i\eta_i[\eta_i - kc_i^T e]\,. \end{aligned}$$

Writing this as a quadratic form in $[e;\, \eta]$ one obtains

$$\dot{V} \leq [e^T\, \eta^T] Y \begin{bmatrix} e \\ \eta \end{bmatrix} < 0\,.$$

This expression is negative $\forall$ nonzero $x,\, z \in \mathbb{R}^n$ if Y is negative definite. ■

Remarks

- The condition for global asymptotic stability of a Lur'e system, stabilized by linear state feedback:

$$\dot{x} = Ax + B\sigma(Cx) + u\,, \quad u = Kx \qquad (7)$$

leads to the same condition as (6), if one takes the Lyapunov function

$$V(x) = x^T P x + \sum_{i}^{n_h} 2\gamma_i \int_0^{c_i^T x} \sigma_i(\rho) k d\rho \qquad (8)$$

with $P = P^T > 0$, $\gamma_i \geq 0$. The same observation of similarity between this stabilization problem and the synchronization problem has been made, e.g. in Wu & Chua [1994].

- For a given Lur'e system and feedback matrix F, one has a linear matrix inequality (LMI) in the unknown matrices P, Γ, T. Finding these unknown matrices corresponds to solving a convex optimization problem [Boyd *et al.*, 1994]. The overall design problem of finding a feedback matrix F together with P, Γ, T such that (6) is satisfied however leads to a nonconvex optimization problem.

3. Example: Synchronization of Chua's Circuit

In order to illustrate the previous Theorem, let us consider the master-slave synchronization problem for Chua's circuit:

$$\begin{cases} (\mathcal{M})\, \dot{x} = A_c x + B_c g(C_c x) \\ (\mathcal{S})\, \dot{z} = A_c z + B_c g(C_c z) + F(x - z) \end{cases} \qquad (9)$$

A Chua's circuit generates the double scroll attractor [Chua *et al.*, 1986] for

$$A_c = \begin{bmatrix} -6.3 & 6.3 & 0 \\ 0.7 & -0.7 & 1 \\ 0 & -7 & 0 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}, \quad C_c = [1 \;\; 0 \;\; 0]$$

and $g(\alpha) = -0.5\alpha - 0.15|\alpha + 1| + 0.15|\alpha - 1|$. Taking $A = A_c$, $B = B_c$, $C = -C_c$ [Guzelis & Chua, 1993] one obtains the representation (2), with $\sigma(\cdot)$ belonging to sector $[0;\, 0.8]$ and $k = 0.8$. In order to find a feedback matrix F and matrices P, Γ and T such that (6) is satisfied and (9) synchronizes, the following constrained nonlinear optimization problem has been solved

$$\min_{F,R,\Gamma,T} \lambda_{\max}(Y) \quad \text{such that} \quad \|F\|_2 \leq c\,, \qquad (10)$$

where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue and $P = R^T R$. The constraint $\|F\|_2 \leq c$ is used because otherwise the norm on F becomes too large. In the simulations $c = 5$ was taken. Sequential quadratic programming (SQP) [Fletcher, 1987] with numerical calculation of the gradients was used in Matlab (function *constr* [The MathWorks Inc., 1994]) in order to minimize (10). The cost function is differentiable as long as the two largest eigenvalues of Y do not coincide [Polak & Wardi, 1982]. Multistart local optimization was done with starting points $\Gamma = 0.1$, $T = 1$, $R = I_3$ and F a random matrix with random elements normally distributed with zero mean and variance 1. Figure 1 shows the behavior of the circuits (9), corresponding to one of the feasible points to (6), for some initial states of x and z.

4. Conclusion

In this paper we investigated the synchronization problem of Lur'e systems, that consist of a linear dynamical system with feedback interconnected to a static nonlinearity that satisfies a sector condition. Using a Lur'e–Postnikov Lyapunov function a matrix inequality is obtained that expresses a sufficient condition for global asymptotic stability of the error system. A systematic design procedure exists in order to find the feedback matrix by solving a constrained nonlinear optimization problem. The method has been demonstrated on Chua's circuit with the double scroll. For secure communication, Lur'e systems that produce high dimensional chaos are needed. Generalized cellular neural networks might be used for that purpose.

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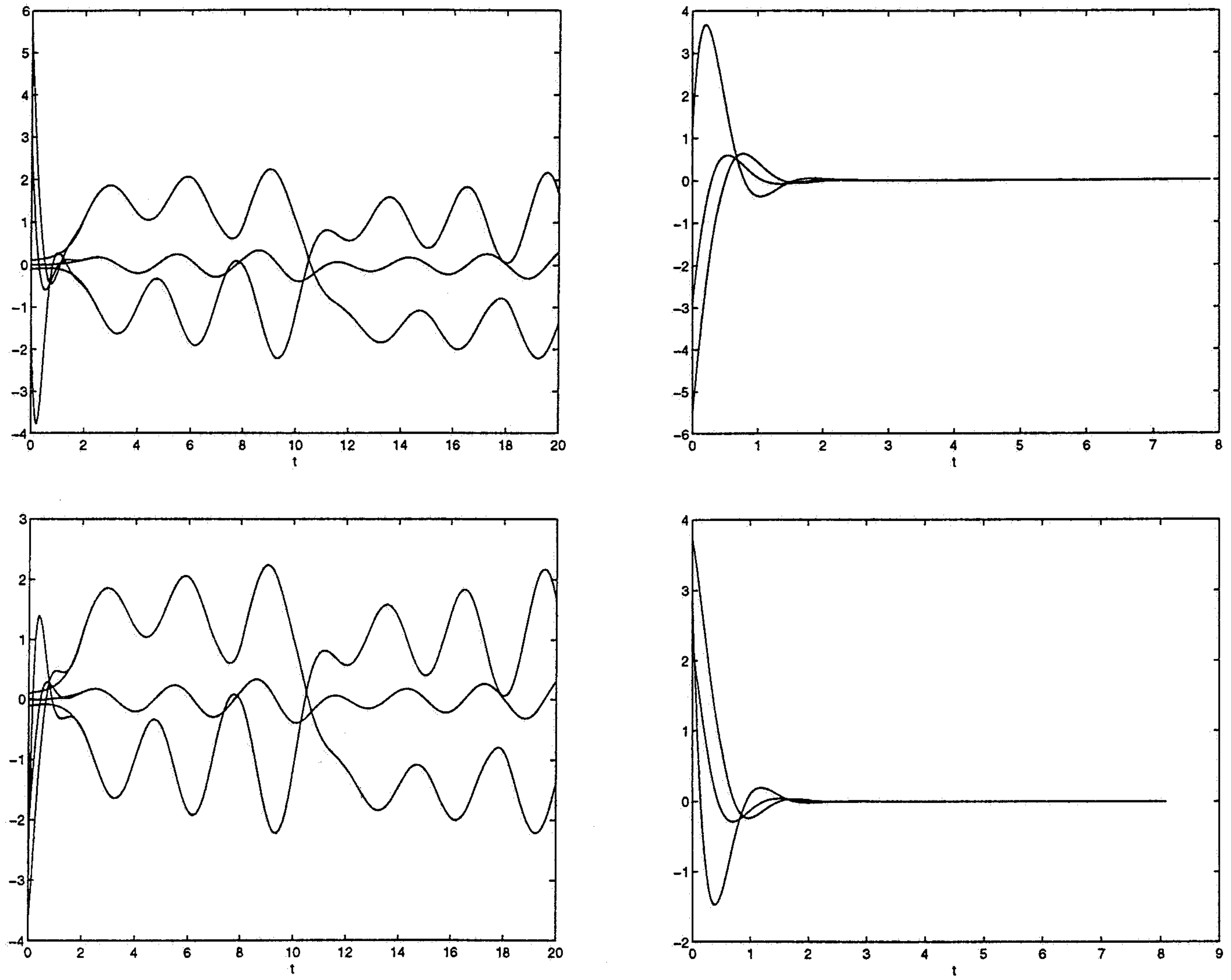

(a) (b)

Fig. 1. (a) The figure shows the 6 state variables through time of a master-slave synchronization system, where the master system behaves as Chua's double scroll for initial state [0.1; 0; −0.1]. For the slave system two randomly chosen initial states were taken, shown at the top and bottom. (b) Corresponding error signals through time.

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References

Boyd, S., El Ghaoui, L., Feron, E. & Balakrishnan V. [1994] *Linear Matrix Inequalities in System and Control Theory*, SIAM (Studies in Applied Mathematics), Vol. **15**.

Chen, G. [1993] "Controlling Chua's global unfolding circuit family," *IEEE Trans. Circ. Syst. I* **40**(11), 829–832.

Chua, L. O., Komuro, M. & Matsumoto T. [1986] "The double scroll family," *IEEE Trans. Circ. Syst. I* **33**(11), 1072–1118.

Fletcher, R. [1987] *Practical Methods of Optimization*, Second Edition (John Wiley and Sons, Chichester and New York).

Guzelis, C. & Chua, L. O. [1993] "Stability analysis of generalized cellular neural networks," *Int. J. Circ. Theor. Appl.* **21**, 1–33.

Hasler, M. [1994], "Synchronization principles and applications," *Circ. Syst.: Tutorials IEEE-ISCAS'94* 314–326.

Khalil, H. K. [1992] *Nonlinear Systems* (Macmillan Publishing Company, New York).

Narendra, K. S. & Taylor, J. H. [1973] *Frequency Domain Criteria for Absolute Stability* (Academic Press, New York).

Polak, E. & Wardi, Y. [1982] "Nondifferentiable optimization algorithm for designing control systems having singular value inequalities," *Automatica* **18**(3), 267–283.

The MathWorks Inc. [1994] *Matlab Version 4.2, Optimization Toolbox.*

Wu, C. W. & Chua, L. O. [1994] "A unified framework for synchronization and control of dynamical systems," *Int. J. Bifurcation and Chaos* 4(4), 979–989.

Wu, C. W. & Chua, L. O. [1995] "Synchronization in an array of linearly coupled dynamical systems," *IEEE Trans. Circ. Syst. I* **42**(8), 430–447.