Relay Feedback Design of Chaotic Systems on Piecewise Linear Feedback

A. Sugiki  S. Hatakemaya  K. Furuta
Department of Applied Systems Engineering
Tokyo Denki University
Ishizaka, Hatoyama, Saitama, Japan
sugiki@hatalab.k.dendai.ac.jp  sho@k.dendai.ac.jp  furuta@k.dendai.ac.jp

Abstract

In this paper, a relay feedback model of Chua’s chaotic system is presented. It is shown that a piecewise-linear feedback element in Chua’s system can be converted to a simple relay and the equivalent relay feedback system is directly designed by applying the bilinear transformation. A feature associated with the relay feedback design is that chaos can be generated for larger sampling-periods compared to the original system in the sampling and zero-order hold discretization.

1 Introduction

Feedback connection of a memoryless nonlinearity with a linear system is recognized as a candidate for chaos-inducing structure, and several types of nonlinearities have been considered [3][4][6][7][9]. Chua’s circuit [9], which is referred to as one of representative chaotic models in the chaos field, also belongs to this category. The Chua circuit consists of a linear system and a piecewise-linear feedback element, and have been analyzed in detail both theoretically and experimentally. On the other hand, the control theory has provided several techniques for analyzing and designing chaotic systems [3][7][8]. As an example, the Chua circuit have been analyzed in terms of parameters providing chaotic motion by the harmonic balance [8]. There is no doubt that results of the control theory is useful for studying chaotic phenomena. From this viewpoint, this paper shows that the piecewise-linear element of Chua’s circuit can be converted into a simple relay. This is achieved by the bilinear transformation, which is an extension of an equivalence between a piecewise-linear feedback system and a relay feedback system provided by Okabayashi and Furuta [5]. We have already shown in [2] a relay feedback system which is equivalent to the Brockett chaotic system [6]. Between the Chua system and the Brockett system, there is some resemblance. Generated trajectories by both systems exhibit a double-scroll type chaotic attractor, and feedback elements have piecewise-linear characteristics. This paper also shows that zeros condition of its linear subsystem in order to generate chaos can be derived by Shilnikov’s theorem [1]. Finally, numerical simulation reveals that the relay feedback design features the generation of chaos for larger sampling-periods compared to the original system in the sampling and zero-order hold discretization.

2 Chua’s Chaotic Circuit

The circuitry of the Chua system is shown in Fig.1.

![Chua's Chaotic Circuit](image)

Figure 1: Chua’s chaotic circuit.

By selecting the state variables $x_1 = v_{C1}$, $x_2 = v_{C2}$, and $x_3 = i_L$, the circuit can be described as the following state-space form:

$$\dot{x} = A_x x + B_x u \quad (1)$$

$$y = C_x x \quad (2)$$

$$u = n_r(y) \quad (3)$$
where
\[
n_c(y) = \begin{cases} 
  m_1 y + (m_1 - m_0) ; & y \geq B_p \\
  m_1 y ; & |y| < B_p \\
  m_0 y - (m_1 - m_0) ; & y \leq -B_p 
\end{cases}
\] (4)

\[
A_r = \begin{bmatrix} 
  -C_1/C_2 & 0 \\
  C_1/C_2 & 0 \\
  0 & 1 \\
\end{bmatrix}
\] (5)

\[
B_r = \begin{bmatrix} 
  1/C_1 \\
  0 \\
\end{bmatrix}^T
\] (6)

\[
C_r = \begin{bmatrix} 
  1 \\
  0 \\
\end{bmatrix}
\] (7)

\[
x = \begin{bmatrix} 
  x_1 \\
  x_2 \\
  x_3 
\end{bmatrix}^T.
\] (8)

The nonlinearity \( n_c \) has the piecewise-linear characteristic as illustrated in Fig.2(a). The state-space model is chaotic when
\[
\begin{align*}
  m_0 &= 0.5, \quad m_1 = 0.8, \quad C_1 = \frac{1}{2}, \quad C_2 = 1 \\
  L &= \frac{1}{4}, \quad G = \frac{1}{2} = 0.68, \quad B_p = 1
\end{align*}
\] (9)

and exhibits the double-scroll type chaotic attractor of Fig.3.

![Figure 2: (a) Nonlinear characteristic in Chua's system, (b) saturation.](image)

3 Conversion of Nonlinear Characteristic in Chua's System

By the transformation illustrated in Fig.4, Chua’s system can be converted to the following feedback system with saturation:
\[
P(s) = \left[ \frac{1 - \delta}{m_1} \right] S(s) \left[ 1 - m_0 S(s) \right]^{-1}
\] (10)

\[
u = -n_{sat}(y)
\] (11)

\[
n_{sat}(y) = \begin{cases} 
  (m_1 - m_0) ; & y \geq \delta \\
  \frac{m_1 - m_0}{\delta} y ; & |y| < \delta \\
  -(m_1 - m_0) ; & y \leq -\delta
\end{cases}
\] (12)

where \( S(s) \) is the transfer function representation of the linear part (1) and (2). Furthermore, letting \( \delta \to 0 \)

\[
P(s) = \left[ \frac{1}{m_1} - S(s) \right] \left[ 1 - m_0 S(s) \right]^{-1}
\] (13)

\[
u = -n_{sat}(y)
\] (14)

\[
n_{sat}(y) = \begin{cases} 
  -(m_1 - m_0) ; & y < 0 \\
  0 ; & y = 0 \\
  m_1 - m_0 ; & y > 0
\end{cases}
\] (15)

The expression (10) can be obtained in the following way. From the conversion illustrated in Fig.4 (b') \rightarrow (c), we have
\[
P(s) = [b_1 - S(s)] [1 - b_2 S(s)]^{-1}.
\] (16)

The conversion illustrated in Fig.4 (a) \rightarrow (h) yields the following relation of signals:
\[
u_{sat} = -u_r + b_1 y_r, \quad y_r = b_2 u_r - y_{sat}.
\] (17)

By substituting \( y_{sat} = [(m_1 - m_0)/\delta] u_{sat} \) and \( y_{sat} = \pm(m_1 - m_0) \) into (17), we have the transformation parameters
\[
b_1 = \frac{1 - \delta}{m_1}, \quad b_2 = m_0.
\] (18)

Therefore, \( P(s) \) and \( G(s) \) can be calculated as follows:

719
\begin{equation}
\begin{align*}
P(s) &= \frac{(1-\delta)C_1C_2Ls^3 + \{(1-\delta)(C_1+C_2)G - m_1C_2Ls^2 + \{(1-\delta)C_1 - m_1LG\}s + (1-\delta)G - m_1\}}{m_1[C_1C_2Ls^3 + \{(C_1+C_2)G - m_0C_2\}Ls^2 + (C_1-m_0LG)s + G - m_0]} \tag{19} \\
G(s) &= \frac{C_1C_2Ls^3 + \{(C_1+C_2)G - m_1C_2Ls^2 + (C_1-m_1LG)s + G - m_1\}}{m_1[C_1C_2Ls^3 + \{(C_1+C_2)G - m_0C_2\}Ls^2 + (C_1-m_0LG)s + G - m_0]} \tag{20}
\end{align*}
\end{equation}

Since \( S(s) \) is

\begin{equation}
S(s) = \frac{C_c(sI-A_c)^{-1}B_c}{C_cC_2Ls^3 + (C_1+C_2)LGs^2 + C_1s + G} \tag{21}
\end{equation}

\begin{equation}
= \frac{C_2Ls^3 + LGs + 1}{C_cC_2Ls^3 + (C_1+C_2)LGs^2 + C_1s + G} \tag{22}
\end{equation}

Letting the (control canonical) realization of \( G(s) \) be \( \{A_1, B_1, C_1, D_1\} \), we have a control input with the implicit form

\begin{equation}
u = -n_{ref}(C_1x + D_1u) \quad (D_1 \neq 0). \tag{23}\end{equation}

In order to derive an explicit form of (23), the usage of the saturation \( n_{sat} \) of Fig (b) in place of \( n_{ref} \) yields

\begin{equation}
u = -\alpha(C_1x + D_1u) \quad (|C_1x + D_1u| < \delta) \tag{24}\end{equation}

\begin{equation}
u = -\alpha(1+\alpha D_1)^{-1}C_1x. \tag{25}\end{equation}

Furthermore, the coordinate is changed using \( T_c \), which is for the control canonical realization of Chua's original system. Namely, the following state model is selected:

\begin{equation}
\dot{x} = (T_cA_1T_c^{-1})x + (T_cB_1)u \tag{26}\end{equation}

\begin{equation}
y = (C_1T_c^{-1})x \tag{27}\end{equation}

\begin{equation}
u = \begin{cases} 
-(m_0 - m_1) : u^* \leq -(m_0 - m_1) \\
|u^*| > m_0 - m_1 & u^* \geq m_0 - m_1 \tag{28}
\end{cases}
\end{equation}

where \( u^* = -\alpha(1+\alpha D_1)^{-1}y \) and \( \alpha = (m_1 - m_0)/\delta \) is assumed to be large enough. Fig. 5 shows the chaotic behavior generated by the above state model.

Thus, numerical simulation also confirms that the Chua system and the relay feedback system are the equivalent chaotic system.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Chaotic behavior from relay feedback: \( x(0) = [0.01, 0.01, 0.1]^T, \alpha = 30, \delta = 0.01. \) \label{fig:chaotic_behavior}}
\end{figure}
4 Analysis of the proposed system

4.1 Eigenvalue Analysis

The piecewise linearity of Chua’s system allows to divide the system into three linear pieces. The closed-loop transfer functions of each piece are

\[ G^+(s) = \frac{N(s)}{(s - \gamma_1)[s - (a_i \pm jb_i)]} \quad (|y| < 1) \]  
\[ G^+(s) = \frac{N(s)}{(s + \gamma_1)[s - (a_i \pm jb_i)]} \quad (1 \geq |y|) \]

where \( N(s) = 0.1429s^2 + 0.0971s + 1 \). This description shows that Chua’s chaos is generated by switching eigenvalues corresponding to these eigenvalues.

Here, the following relation concerning the eigenvalues holds:

\[ G(s) = \frac{1.25(s - \gamma_1)[s - (a_i + jb_i)]}{(s - \gamma_1)[s - (a_i \pm jb_i)]} \]

This relation is confirmed by considering the situation from the root locus.

Then, consider the simplified system

\[ G^+(s) = \frac{(s - \gamma_1)[s - (a_i \pm jb_i)]}{(s - \gamma_1)[s - (a_i \pm jb_i)]} \]

where \( f: R^d \rightarrow R^d \) is continuous and piecewise-linear. Let the origin be an equilibrium with a real eigenvalue \( \gamma > 0 \) and a complex conjugate pair \( \beta \pm j\omega \) \( (\beta < 0, \omega \neq 0) \). If

(a) \( |\beta| < \gamma \), and

(b) there is a homoclinic orbit through the origin.

Here, we are considering the condition (a). Regarding relay feedback (discontinuous) as saturation feedback with large enough slope (continuous), our relay feedback design allows to verify the condition (a) from the linear part \( G(s) \) only, which can be extended to an assertion for chaos synthesis on a biproper linear system and relay feedback structure.

5 Chua’s System and Sampling

For Chua’s system and its relay feedback model, we are considering the sampling and zero-order hold discretization

\[ x[(k + 1)T] = \Phi x(kT) + \Gamma u(kT) \]
\[ y(kT) = C x(kT) + Du(kT) \]

where \( \Phi = e^{AT} \), \( \Gamma = e^{AT} \int_0^T e^{-A\tau} d\tau B \), and \( T \) is the sampling period. The Chua system bifurcates with increasing \( T \) as shown in Fig.6.

![Figure 6: Bifurcation diagram of Chua’s system](image)

Chua’s original system exhibits the double-scroll attractor for sufficiently small \( T \). Its behavior is bifurcated into the spiral type of Fig.7 (above) with increasing \( T \). For larger \( T \), a limit cycle appears.
that the proposed relay feedback design can reduce the sampling effect for the existence of chaotic nature. This is considered to be a feature of the transformation to the relay feedback.

6 Conclusion

This paper has presented a relay feedback chaotic system which is equivalent to Chua’s circuit, and simplified the chaos-inducing mechanism in terms of the nonlinear feedback characteristic. The features of the relay feedback system include that the sampling effect can be reduced. This paper also showed that, for chaos synthesis using a biproper linear system and relay feedback, zeros condition of its linear subsystem is connected to Shilnikov’s theorem (a) with using the relay feedback property.

References


