Chaos from a Time-Delayed Chua's Circuit

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Invited Paper

Abstract—**By replacing the parallel LC "resonator" in Chua's circuit by a lossless transmission line that is terminated by a short circuit, we obtain a "time-delayed Chua's circuit," whose time evolution is described by a pair of *linear partial* differential equations with a *nonlinear* boundary condition. If we neglect the capacitance across the Chua's diode, which is described by a nonsymmetric piecewise-linear $v_R - i_R$ characteristic, the resulting idealized "time-delayed" Chua's circuit is described *exactly* by a *scalar nonlinear difference equation* with continuous time, which makes it possible to characterize its associated nonlinear dynamics and spatial chaotic phenomena.**

From a mathematical view point, circuits described by *ordinary* differential equations can generate only *temporal* chaos, whereas the time-delayed Chua's circuit can generate *spatial-temporal* chaos. Except for stepwise periodic oscillations, the *typical solutions* of the idealized time-delayed Chua's circuit consist of either *weak turbulence* or *strong turbulence*, which are examples of "ideal" (or "dry") turbulence. In both cases, we can observe infinite processes of spatial-temporal coherent structure formations.

Under *weak turbulence*, the graphs of the solution tend to limit sets that are *fractals* with a Hausdorff dimension between 1 and 2 and is therefore larger than the topological dimension (of sets).

Under *strong turbulence*, the "limit" oscillations are oscillations whose amplitudes are *random functions*. This means that the attractor of the idealized time-delayed Chua's circuit already contains random functions, and spatial self-stochasticity phenomenon can be observed.

I. Generalizing Chua's Circuit to Infinite Dimensions

The original Chua's circuit [1] consists of a *linear passive* resonator (parallel LC tune circuit) connected across a *nonlinear active* circuit composed of a Chua's diode [2], a linear capacitor C_1, and a linear resistor R, as shown in Fig. 1.

In most publications [3], Chua's diode is characterized by a *continuous odd-symmetric* three-segment piecewise-linear function. In this paper, we consider an "infinite-dimensional" generalization of Chua's circuit obtained by replacing the LC resonant circuit by a *lossless* transmission line of length l terminated on its left ($x = 0$) by a *short circuit*, as shown in Fig. 2(a).

Since the effect of the transmission line is to provide a "time delay" in the dynamics of Chua's circuit, we will henceforth call this circuit the "time-delayed Chua's circuit." Our main objective in this paper is to investigate the asymptotic behaviors of the *voltage* $v(x,t)$ and the *current* $i(x,t)$ at all points along the transmission line ($0 \leq x \leq l$). Since the most interesting and complex "turbulent-like" dynamical behaviors occur when the $v_R - i_R$ characteristic of Chua's diode is *not* symmetric with respect to the origin, we have chosen the three-segment piecewise-linear function shown in Fig. 2(b), where $i_R = G(v_R - E)$.

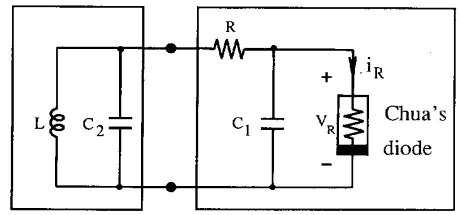

Fig. 1. Decomposition of Chua's circuit into a passive linear subcircuit on the left and an active nonlinear subcircuit on the right.

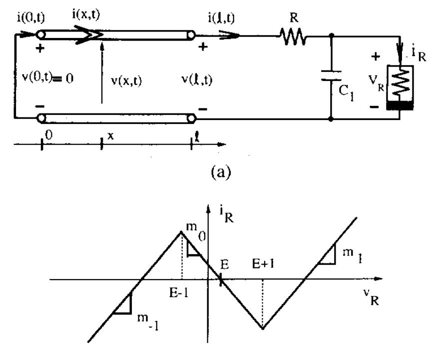

Fig. 2. (a) Time-delayed Chua's circuit; (b) $v_R - i_R$ characteristic of Chua's diode is chosen to be a three-segment piecewise-linear function symmetric ($m_{-1} = m_1$) with respect to the point $v_R = E > 0$.

II. Time Evolution Equations

The lossless transmission line in Fig. 2(a) is defined by the following linear partial differential equations:

$$\frac{\partial v(x,t)}{\partial x} = -L\frac{\partial i(x,t)}{\partial t} \quad (1)$$

$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial v(x,t)}{\partial t} \quad (2)$$

where L and C denote the inductance and capacitance per unit length of the transmission line. The boundary conditions are given respectively at $x = 0$, and $x = 1$ by

$$v(0,t) = 0 \quad (3)$$

$$i(l,t) = G(v(l,t) - E - Ri(l,t)) + C_1\frac{\partial(v(l,t) - Ri(l,t))}{\partial t} \quad (4)$$

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where $G(\cdot)$ is defined by

$$G(u) = \begin{cases} m_0 u, & |u| \leq 1 \\ m_1 u - (m_1 - m_0) sgn\ u, & |u| \geq 1 \end{cases} \quad (5)$$

and $u = v_R - E$.

The general solution of (1) and (2) has the form

$$v(x,t) = \alpha(t - \frac{x}{\nu}) - \alpha(t + \frac{x}{\nu}) \quad (6)$$

$$i(x,t) = \frac{1}{Z}[\alpha(t - \frac{x}{\nu}) + \alpha(t + \frac{x}{\nu})] \quad (7)$$

where $\nu = \sqrt{LC}$ is the velocity of the incident and reflected waves, and $Z = \sqrt{\frac{L}{C}}$ is the characteristic impedance of the transmission line.

Note that (1)–(4) constitute a system of two *linear* partial differential equations with a *nonlinear* boundary condition (at $x = l$). Since this problem is presently analytically intractable, we will investigate here only the limiting case where the capacitance C_1 in Fig. 1(a) is replaced by an open circuit, i.e., $C_1 \to 0$. Under this assumption, we can substitute (6)–(7) into (4) with $C_1 = 0$ and introduce the new variables

$$\tau = \frac{\nu}{2l}t - \frac{1}{2}, \quad \beta(\tau) = \alpha\left(\frac{2l}{\nu}\tau\right) \quad (8)$$

to obtain the following *difference equation*:

$$\beta(\tau + 1) = f(\beta(\tau)). \quad (9)$$

The symbol $f(\cdot)$ denotes a piecewise-linear (single-valued or multivalued) function defined by

$$f(\beta) = A_k\beta - B_k, \text{ where } \beta \in I_k, \quad k = 0, \pm 1 \quad (10)$$

$$\begin{aligned} A_k &= -1 + q_k \\ B_k &= \frac{q_0}{2}\left[E + k\left(1 - \frac{m_0}{m_k}\right)\right] \\ q_k &= \frac{2Z}{\frac{1}{m_k} + R + Z} \\ m_{-1} &= m_{+1} \\ I_0 &= \{\beta : \left|\beta - \frac{E}{2}\right| \leq \delta\} \\ I_{\pm 1} &= \{\beta : \pm(\beta - \frac{E}{2}) > \delta\} \\ \delta &= \frac{m_0 Z}{q_0}. \end{aligned} \quad (11)$$

It follows from the above derivations that the time evolution of the "time-delayed Chua's circuit with $C_1 = 0$ is governed by a *scalar* nonlinear difference equation with a continuous argument, namely, (9). The qualitative behavior of this equation is determined by the properties of the 1-D map

$$\boxed{\beta \mapsto f(\beta)} \quad (12)$$

where $f(\cdot)$ is defined in (10).

III. Asymptotic Behaviors

To describe the behaviors of the solutions of the "time-delayed Chua's circuit with $C_1 = 0$, it will be convenient for us to use the language of dynamical systems theory. Let $X = [0, l]$, and let $C(X)$ denote the space of all smooth or continuous functions

$$\{(v(x), i(x)) : 0 \leq x \leq l\} \quad (13)$$

with some appropriate topology, e.g., uniform metric. The solution $v_t(x) = v(x,t)$ and $i_t(x) = i(x,t)$ of (1)–(4) (assuming $C_1 = 0$) with initial condition $v_0(x) = v(x,0)$ and $i_0(x) = i(x,0)$ defines an *infinite-dimensional dynamical system*

$$\boxed{\mathcal{F}^t : (v_0(x), i_0(x)) \mapsto (v_t(x), i_t(x)).} \quad (14)$$

For the class of problems defined above, it is the usual situation that the trajectories $\mathcal{F}^t[(v_0, i_0]$ have no ω limit set in the phase space $C(X)$, and the dynamical system (14) has no attractor in the phase space $C(X)$ (at least for the reason that the Lipschitz constants can grow to ∞). In order to describe the *asymptotic* (as $t \to \infty$) behavior of the infinite-dimensional system (14), it is necessary for us to *complete* the phase space $C(X)$ with the help of a suitable metric. For "weak" turbulent oscillations to be described below, it is sufficient to use the *Hausdorff* metric for the function graphs. For "strong" turbulent oscillations, however, we need to introduce some specially constructed metric, such as the one given in [4] involving all finite-dimensional joint distributions in combination with the operation of averaging. We will henceforth assume that our phase space has been *completed* by compactification via some special metric. In the following, it will be useful to consider both the 1-D system (12) and the infinite-dimensional system (14) simultaneously.

Case 1. $R > -\frac{1}{m_0} - Z$: In this case, the behavior of both dynamical systems are simple. In the *1-D system*, there exists an attracting *fixed point*, or attracting *cycle*, that attracts all, or almost all, trajectories. Correspondingly, in the *infinite-dimensional* system, the oscillations either vanish as $t \to \infty$ or tend to a periodic solution, as in Witt [5] and Nagumo and Shimura [6].

Case 2. $R < -\frac{1}{m_0} - Z$: In this case, it is possible to observe *chaotic oscillations* in the *infinite-dimensional* system. The system can also have oscillations that vanish as $t \to \infty$ as well as stable *stepwise* periodic oscillations but only with period 2 (for example, when $R = 0$ or $R \approx 0$). In addition, the infinite-dimensional system can also have a more complicated asymptotically stable periodic solution of period 4. The corresponding solutions for the 1-D system are as follows. The 1-D map f has an attracting fixed point, an attracting cycle of period 2, or an attracting cycle of period 4, which occurs after a period-doubling bifurcation.

If we exclude the above classes of solutions, where it is possible to derive exact estimates, then only two other types of asymptotic behaviors can exist for the infinite-dimensional system (14), namely, *weak turbulent oscillations* or *strong turbulent oscillations* [7], [8].

The simpler case of *weak turbulence* corresponds to the case where the 1-D map $f(\cdot)$ has an attracting cycle that attracts all trajectories except those belonging to the repeller, which is a *Cantor set* K consisting of unstable trajectories. It has a zero Lebesque measure (mes $K = 0$) but a positive Hausdorff dimension ($dim_H > 0$). If "p" is the period of a stable cycle of the *1-D* system, then almost every trajectory of the *infinite-dimensional* system (i.e., almost all solutions of (1)–(4) with $C_1 = 0$) will be asymptotically periodic with period p, and its ω-limit set is a *periodic* trajectory in the completed phase space. Each point of the limit trajectory is a multivalued (on some Cantor set) function graph that is a fractal set with a Hausdorff dimension $dim_H > 1$. The attractor of the *infinite-dimensional* system in this case consists of such periodic orbits of the same period.

The most complicated case of *strong turbulence* corresponds to the situation where the *1-D* map $f(\cdot)$ has no attracting cycles, and its attractor consists of one or several intervals where there is a smooth invariant measure μ. In this case, if we use the special metric defined in [4], then the following asymptotic behavior applies:

The ω-limit set of almost every trajectory of the infinite-dimensional system (14) is a periodic trajectory where each point on the trajectory is a *random* function. The distribution of the values of such a random function for each x is determined by the measure μ. Hence, in this case, the *attractor of the infinite-dimensional system consists of random functions*, which form periodic orbits. This phenomenon corresponds to a very strong "spatial chaos."

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