USING DYNAMIC NEURAL NETWORKS TO GENERATE CHAOS: AN INVERSE OPTIMAL CONTROL APPROACH

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This Letter suggests a new approach to generating chaos via dynamic neural networks. This approach is based on a recently introduced methodology of inverse optimal control for nonlinear systems. Both Chen's chaotic system and Chua's circuit are used as examples for demonstration. The control law is derived to force a dynamic neural network to reproduce the intended chaotic attractors. Computer simulations are included for illustration and verification.

1. Introduction

Control methods for general nonlinear systems have been extensively developed since the early 1980s. A special technique, to be employed in this paper, is the “inverse optimal control” method. This method was recently developed based on the input-to-state stability concept [Kristic & Deng, 1998]. Here, we show that this methodology can be applied to control a dynamic neural network to produce some complex nonlinear behaviors such as chaos from a given model.

It is now known that a neural network, if appropriately designed, can produce chaos after intensive learning [Chen *et al.*, 1997]. Small-amplitude perturbations can also achieve the goal of chaos generation [Wang & Chen, 2000]. Yet, how to achieve this goal by using some traditional feedback control strategies is still open for investigation. This Letter addresses this issue and completes a conventional controller design for chaos generation.

Briefly, in this Letter, a new approach is developed for producing chaos via a dynamic neural network from an inverse optimal control approach. Two representative examples of a smooth Chen's chaotic system and a piecewise continuous Chua's circuit are used for demonstration. An effective control law is derived to force the neural network to produce the intended chaotic attractors. Computer simulations are included to show the success and effectiveness of the design.

2. Mathematical Description

Consider a dynamic neural network in the following form:

$$\dot{x} = Ax + Wf(x) + u, \quad x, u \in \mathbb{R}^n, \quad A, W \in \mathbb{R}^{n \times n} \quad (1)$$

where x is the state, u is the input, $A = -\lambda I$, with λ being a positive constant, is the state-feedback

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matrix, $f(\cdot)$ is a bounded sector function [Khalil, 1996] such that $f(x) = 0$ only at $x = 0$ and $\lim_{x \rightarrow \infty} (f(x))^\top x = +\infty$, and W is the weight matrix.

A common selection for the basic element of $f(\cdot)$ is the sector function $\sigma(\cdot) = \tanh(\cdot)$. There exist positive constants k_1 and k_2 such that $k_1 \|x\|_2^2 \leq (f(x))^\top x \leq k_2 \|x\|_2^2$. Also, (1) can be expressed as $\dot{x} = \tilde{f}(x) + g(x)u$, with $\tilde{f}(x) = Ax + Wf(x)$ and $g(x) = I$. It is clear that $x = 0$ is an equilibrium point of this system, when $u = 0$.

Let the following be the model system for the neural network to track:

$$\begin{aligned} \dot{x}_r &= f(x_r) + g(x_r)u_r, \\ x_r, u_r &\in \mathbb{R}^n, f(\cdot) \in \mathbb{R}^n, g(\cdot) \in \mathbb{R}^{n \times n} \end{aligned} \quad (2)$$

where x_r is the state, u_r is the input, and $f(\cdot)$ and $g(\cdot)$ are smooth nonlinear functions.

It is clear that this setup is very general, and the model (2) can be a complex nonlinear model such as a chaotic system.

3. Model Following as a Stabilization Problem

Define the model following error by

$$e = x - x_r \quad (3)$$

and then substitute (1) and (2) into (3), with $f(\cdot) := \sigma(\cdot) = \tanh(\cdot)$. Then, we obtain

$$\dot{e} = Ae + W\sigma(e + x_r) + u - f(x_r) - g(x_r)u_r + Ax_r \quad (4)$$

Adding to and subtracting from (4) the terms $W\sigma(x_r)$ and $\alpha(t)$, we have

$$\begin{aligned} \dot{e} &= Ae + W(\sigma(e + x_r) - \sigma(x_r)) + Ax_r + W\sigma(x_r) \\ &\quad + \alpha(t) - (f(x_r) + g(x_r)u_r) + (u - \alpha(t)) \end{aligned} \quad (5)$$

where $\alpha(t)$ is a function to be determined.

For system (1) to follow model (2), the following natural solvability assumption is needed (see [Kristic & Deng, 1998] for a reason):

Assumption 1. There exist functions $\rho(t)$ and $\alpha(t)$ such that

$$\begin{aligned} \frac{d\rho(t)}{dt} &= A\rho(t) + W\sigma(\rho(t)) + \alpha(t) \\ \rho(t) &= x_r(t) \end{aligned} \quad (6)$$

Then, it follows from (6) and (2) that

$$Ax_r + W\sigma(x_r) + \alpha(t) = f(x_r) + g(x_r)u_r \quad (7)$$

Consequently, (5) becomes

$$\dot{e} = Ae + W(\sigma(e + x_r) - \sigma(x_r)) + (u - \alpha(t)) \quad (8)$$

Next, introduce the following functions:

$$\begin{aligned} \phi(e, x_r) &= \sigma(e + x_r) - \sigma(x_r) \\ \tilde{u} &= (u - \alpha(t)) \end{aligned}$$

We then rewrite (8) as

$$\dot{e} = Ae + W\phi(e, x_r) + \tilde{u} \quad (9)$$

where x_r is considered as an external disturbance input. It is clear that $e = 0$ is an equilibrium point of (9), when $\tilde{u} = 0$.

Consider the function $\phi(e, x_r) = \sigma(e + x_r) - \sigma(x_r)$. Clearly, if $e = 0$ then $\phi(e, x_r) = 0$. Moreover, for each component, $e_i > 0$ implies $e_i + x_{ri} > x_{ri}$ for all x_{ri} . Since σ is monotonically increasing, $\sigma(e + x_r) > \sigma(x_r)$ and $\phi(e, x_r)^\top e = (\sigma(e + x_r) - \sigma(x_r))^\top e > 0$. Similarly, $e_i < 0$ implies $e_i + x_{ri} < x_{ri}$ for all x_{ri} . Since σ is monotonically increasing, $\sigma(e + x_r) < \sigma(x_r)$ and $\phi(e, x_r)^\top e = (\sigma(e + x_r) - \sigma(x_r))^\top e > 0$. Therefore, $\phi(e, x_r)$ is a sector function and is Lipschitz with respect to e .

To this end, the model following problem can be restated as a global asymptotical stabilization problem for system (9).

3.1. Inverse optimal control

To globally and asymptotically stabilize system (9), we proceed along the line of the inverse optimal control approach [Kristic & Deng, 1998], as follows.

First, we find a candidate function as an input-to-state control Lypaunov function. This is essential for the design of a globally and asymptotically stabilizing control law in this approach. We choose

$$V(e) = \sum_{i=1}^n \int_0^{e_i} \phi(\eta, x_r) d\eta \quad (10)$$

Since $\phi(e, x_r)$ is a sector function with respect to e , $V(e)$ is radially unbounded, namely, $V(e) > 0$

for all $e \neq 0$, and $V(e) \rightarrow +\infty$ as $e \rightarrow \infty$. Its time-derivative can be computed as follows:

$$\begin{aligned}\dot{V}(e) &= \phi(e, x_r)^\top (Ae + W\phi(e, x_r) + \tilde{u}) \\ &= \phi(e, x_r)^\top Ae + \phi(e, x_r)^\top W\phi(e, x_r) \\ &\quad + \phi(e, x_r)^\top \tilde{u} \\ &= -\lambda\phi(e, x_r)^\top e + \phi(e, x_r)^\top W\phi(e, x_r) \\ &\quad + \phi(e, x_r)^\top \tilde{u} \\ &:= L_f V + (L_g V) \tilde{u}\end{aligned}\quad (11)$$

where $L_f V = -\lambda\phi(e, x_r)^\top e + \phi(e, x_r)^\top W\phi(e, x_r)$ and $L_g V = \phi(e, x_r)^\top$. It should be noted that $\partial V / \partial x_r = 0$.

Next, consider the following inequality, proved in [Sanchez & Perez, 1999]:

$$X^\top Y + Y^\top X \leq X^\top \Lambda X + Y^\top \Lambda^{-1} Y \quad (12)$$

which holds for all matrices $X, Y \in \mathbb{R}^{n \times k}$ and $\Lambda \in \mathbb{R}^{n \times n}$ with $\Lambda = \Lambda^\top > 0$.

Applying (12) with $\Lambda = I$ to $\phi(e, x_r)^\top W\phi(e, x_r)$, we obtain

$$\begin{aligned}\dot{V}(e) &\leq -\lambda\phi(e, x_r)^\top e + \frac{1}{2}\phi(e, x_r)^\top \phi(e, x_r) \\ &\quad + \frac{1}{2}\phi(e, x_r)^\top W^\top W\phi(e, x_r) + \phi(e, x_r)^\top \tilde{u}\end{aligned}\quad (13)$$

Again, since $\phi(e, x_r)$ is a sector function with respect to e , there exist positive constants k_1 and k_2 such that $k_1 \|e\|_2^2 \leq \phi(e, x_r)^\top e \leq k_2 \|e\|_2^2$. Also, since $\phi(e, x_r)$ is Lipschitz with respect to e , there exists a positive constant L_ϕ such that $\phi(e, x_r)^\top \phi(e, x_r) \leq L_\phi^2 \|e\|_2^2$. Therefore, (13) can be rewritten as

$$\begin{aligned}\dot{V} &\leq -\left(\lambda k_1 - \frac{1}{2}L_\phi^2\right) \|e\|_2^2 \\ &\quad + \frac{1}{2}\phi(e, x_r)^\top W^\top W\phi(e, x_r) + \phi(e, x_r)^\top \tilde{u}\end{aligned}\quad (14)$$

Now, we suggest to use the following control law:

$$\tilde{u} = -(W^\top W + I)\phi(e, x_r) := -\beta(R(e))^{-1}(L_g V)^\top \quad (15)$$

Here, β is a positive constant and $(R(e))^{-1}$ is a function of e in general, but for the current purpose it is chosen as

$$(R(e))^{-1} = \frac{1}{\beta}(W^\top W + I) \quad (16)$$

The motivation for this choice of the control law will be seen from the optimization problem discussed below.

At this point, substituting (15) into (14), we obtain

$$\begin{aligned}\dot{V}(e) &\leq -\left(\lambda k_1 - \frac{1}{2}L_\phi^2\right) \|e\|_2^2 \\ &\quad - \frac{1}{2}\|W^\top W\|L_\phi^2 \|e\|_2^2 - L_\phi^2 \|e\|_2^2 \\ &\leq -\left(\lambda k_1 - \frac{1}{2}L_\phi^2 + \frac{1}{2}\|W^\top W\|L_\phi^2 + L_\phi^2\right) \|e\|_2^2\end{aligned}\quad (17)$$

If $\lambda k_1 - (1/2)L_\phi^2 + (1/2)\|W^\top W\|L_\phi^2 + L_\phi^2 > 0$, or equivalently, $\|W^\top W\| > 2(-\lambda k_1 - (1/2)L_\phi^2)/L_\phi^2$ which can always be satisfied, then $\dot{V}(e) < 0$ for all $e \neq 0$. This means that the proposed control law (15) can globally and asymptotically stabilize system (9), thereby ensuring the model following of (2) by (1).

It should be noted that system (9) is input-to-state stabilizable. This is because its control Lyapunov function satisfies the ‘‘small control property’’ [Kristic & Deng, 1998]. Besides, the inverse optimal control problem, defined below, is indeed solvable, and so Assumption 1 above is satisfied.

To assign a control gain to the control law, following [Kristic & Deng, 1998], we consider (15) and define a cost functional by

$$J(\tilde{u}) = \lim_{t \rightarrow \infty} \left\{ 2\beta V(e) + \int_0^t (l(e) + \tilde{u}^\top R(e)\tilde{u}) d\tau \right\} \quad (18)$$

where

$$\begin{aligned}l(e) &= -2\beta L_f V + 2\beta(L_g V)(R(e))^{-1}(L_g V)^\top \\ &\quad + \beta(\beta - 2)(L_g V)(R(e))^{-1}(L_g V)^\top \\ &= -2\beta L_f V + \beta(L_g V)(\beta(R(e))^{-1})(L_g V)^\top\end{aligned}$$

The basic idea of the inverse optimal control theory is to require that $l(e)$ be radially unbounded, i.e. $l(e) > 0$ for all $e \neq 0$ and $l(e) \rightarrow +\infty$ as $e \rightarrow \infty$. To prove this property, we first specify the term $\beta(R(e))^{-1}(L_g V)^\top$ in the expression of $l(e)$, to be $(W^\top W + I)\phi(e, x_r)$ (see (16)). We also specify the terms $L_g V$ and $L_f V$ by their definitions. We thus

obtain

$$l(e) = 2\beta\lambda\phi(e, x_r)^\top e - 2\beta\phi(e, x_r)^\top W\phi(e, x_r) + \beta\phi(e, x_r)^\top (W^\top W + I)\phi(e, x_r) \quad (19)$$

Next, by applying inequality (12) to the second term of the right-hand side of (19), we obtain

$$\begin{aligned} l(e) &\geq 2\beta\lambda\phi(e, x_r)^\top e - \beta\phi(e, x_r)^\top \phi(e, x_r) \\ &\quad - \beta\phi(e, x_r)^\top W^\top W\phi(e, x_r) \\ &\quad + \beta\phi(e, x_r)^\top (W^\top W + I)\phi(e, x_r) \\ &\geq 2\beta\lambda\phi(e, x_r)^\top e \end{aligned}$$

Since $\phi(e, x_r)^\top$ is a sector function with respect to e , $l(e) \geq 2\beta\lambda\phi(e, x_r)^\top e \geq 0$ and $\lim_{e \rightarrow \infty} l(e) = +\infty$, which satisfies the radially unbounded condition.

Then, substitute the term $\tilde{u}$, defined by (15), into (11). We obtain

$$\dot{V} = L_f V + (L_g V)(-\beta(R(e))^{-1})(L_g V)^\top$$

Multiplying it by -2β , we obtain

$$-2\beta\dot{V} = -2\beta L_f V + 2\beta^2(L_g V)(R(e))^{-1}(L_g V)^\top$$

Finally, taking into account (15), which implies

$$\tilde{u}^\top R(e)\tilde{u} = \beta^2(L_g V)(R(e))^{-1}(L_g V)^\top$$

we arrive at

$$l(e) + \tilde{u}^\top R(e)\tilde{u} = -2\beta\dot{V} \quad (20)$$

To this end, substituting (20) into (18), we have

$$\begin{aligned} J(\tilde{u}) &= \lim_{t \rightarrow \infty} \{2\beta V(e(t)) + \int_0^t -2\beta\dot{V} d\tau\} \\ &= \lim_{t \rightarrow \infty} \{2\beta V(e(t)) - 2\beta V(e(t)) + 2\beta V(e(0))\} \\ &= 2\beta V(e(0)) \end{aligned}$$

Thus, the minimum of the cost functional is $J(\tilde{u}) = 2\beta V(e(0))$, for the optimal control law (15).

To summarize, the optimal and stabilizing control law, which guarantees the model following requirement of (2) followed by (1), is given by

$$\tilde{u}^* = -(W^\top W + I)\phi(e, x_r)$$

In order to obtain the final controller u precisely, which will then be input to the neural network (1), we take into account the equalities $u =$

$\tilde{u}^* + \alpha(t)$ and $\alpha(t) = f(x_r) + g(x_r)u_r - Ax_r - W\sigma(x_r)$. This leads to

$$\begin{aligned} u &= -W^\top W\phi(e, x_r) - \phi(e, x_r) + f(x_r) \\ &\quad + g(x_r)u_r - Ax_r - W\sigma(x_r) \end{aligned} \quad (21)$$

Substituting (21) into (1) then gives

$$\begin{aligned} \dot{x} &= Ax + W\sigma(x) - W^\top W\phi(e, x_r) - \phi(e, x_r) \\ &\quad + f(x_r) + g(x_r)u_r - Ax_r - W\sigma(x_r) \\ &= -\lambda e + W(\sigma(e + x_r) - \sigma(x_r)) - W^\top W\phi(e, x_r) \\ &\quad + I\phi(e, x_r) + f(x_r) + g(x_r)u_r \\ &= -\lambda e + (W - W^\top W + I)(\sigma(e + x_r) - \sigma(x_r)) \\ &\quad + f(x_r) + g(x_r)u_r \end{aligned} \quad (22)$$

It follows from the last equation of (22) that, as $e \rightarrow 0$, the desired model following goal is achieved.

4. Chaos Production

This section demonstrates the applicability of the proposed approach to a chaos production problem. Chaos has been shown to be quite useful in many engineering applications, and there is a strong and increasing demand for generating chaos at will [Chen & Dong, 1998]. In this new research direction, chaos production is one important task, often using neural networks [Chen *et al.*, 1997; Chen & Dong, 1998].

To show that the model following controller designed above can also accomplish this chaos production task, we apply the developed methodology to a recently discovered chaotic system [Chen & Ueta, 1999], referred to as Chen's system by other authors, and the familiar Chua's circuit. The first system is described by

$$\begin{aligned} \dot{x}_r &= a(y_r - x_r) \\ \dot{y}_r &= (c - a)x_r - x_r z_r + cy_r \\ \dot{z}_r &= x_r y_r - bz_r \end{aligned} \quad (23)$$

which has a chaotic attractor as shown in Fig. 1 when $a = 35$, $b = 3$, $c = 28$. It has been experienced that this chaotic system is relatively difficult to control as compared to the Lorenz and Chua's systems due to its prominent three-dimensional and some complex features.

Fig. 1. Chen's chaotic attractor.

To follow the chaotic attractor of system (23), we select the following controlled neural network:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0.3 & 0.8 & 0 \\ 0.4 & 0.3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tan h(x) \\ \tan h(y) \\ \tan h(z) \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (24)$$

It can be easily verified that network (24) satisfies the condition to be input-to-state stable; actually, it is even globally asymptotically stable. Nevertheless, the proposed controller can introduce chaos into the network.

Chua's circuit, on the other hand, is shown in Fig. 2. This chaotic circuit consists of only one inductor (L), two capacitors (C_1 , C_2), one linear resistor (R), and one piecewise-linear resistor (g).

Fig. 2. Chua's circuit.

The dynamical equation of Chua's circuit is described by

$$\begin{cases} C_1 \dot{v}_{C1} = \frac{1}{R}(v_{C2} - v_{C1}) - g(v_{C1}) \\ C_2 \dot{v}_{C2} = \frac{1}{R}(v_{C1} - v_{C2}) + i_L \\ L \dot{i}_L = -v_{C2} \end{cases} \quad (25)$$

where i_L is the current through the inductor L , v_{C1} and v_{C2} are the voltages across C_1 and C_2 , respectively, and

$$g(v_{C1}) = m_0 v_{C1} + \frac{1}{2}(m_1 - m_0)(|v_{C1} + 1| - |v_{C1} - 1|)$$

with $m_0 < 0$ and $m_1 < 0$ being some appropriately chosen constants. This piecewise-linear function is shown in Fig. 3 for clarity.

By defining $p = C_2/C_1 > 0$ and $q = C_2 R^2/L > 0$, and changing the variables

$$\begin{aligned} x(\tilde{t}) &= v_{C1}(t), & y(\tilde{t}) &= v_{C2}(t), \\ z(\tilde{t}) &= R i_L(t), & \tilde{t} &= \frac{t}{(C_2 R)}, \end{aligned}$$

Fig. 3. The piecewise-linear resistance in Chua's circuit.

Fig. 4. Chaotic trajectories of Chua's circuit.

the above circuit equations can be reformulated in the following canonical (dimensionless) form:

$$\begin{cases} \dot{x} = p(-x + y - f(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -qy, \end{cases} \quad (26)$$

where $f(x) = Rg(v_{C_1})$.

Figure 4 shows a double scroll attractor of the circuit, generated with $p = 10.0$, $q = 14.87$, $m_0 = -0.68$, $m_1 = -1.27$, and initial conditions $(-0.1, -0.1, -0.1)$.

4.1. Simulation results

For the network (24) to follow the chaotic system (23), we implement the control law (21). In our simulation, the following initial conditions were used:

$$\begin{pmatrix} x_r(0) \\ y_r(0) \\ z_r(0) \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ 37 \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 70 \\ 90 \end{pmatrix}$$

Our simulation produced a three-dimensional Chen's attractor that is visually indistinguishable from that shown in Fig. 1. Therefore, to reveal more insights of the reproduced attractor, Fig. 5 shows the resulting orbit for the first state variable, while those corresponding to the second and the third state variables are displayed in Figs. 6 and 7, respectively.

Fig. 5. First state variable produced.

Fig. 6. Second state variable produced.

Fig. 7. Third state variable produced.

Fig. 8. Chua's chaotic attractor produced.

Finally, the chaos production simulation for Chua's circuit is similarly carried out, with initial condition $(x(0), y(0), z(0)) = (-5, 5, 5)$. Again, the resulting figure is seemingly no different from the original one, so to visualize the production process we show the forming chaotic attractor in Fig. 8.

It should be emphasized that both Chen's and Chua's chaotic attractors were produced by using the same neural network and the proposed control method.

5. Conclusions

We have presented a new controller designed for model following of a general nonlinear system. This framework is based on the dynamic neural networks and the methodology is based on the inverse optimal control approach. The proposed control scheme is applied to the production of chaotic attractors, for Chen's system and Chua's circuit, with success. Further research is undertaken to extend this approach to robust adaptive tracking control for nonlinear complex dynamical systems, along the line of the studies given in [Poznyak *et al.*, 1999; Sanchez & Perez, 1999].

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