Equation (3) is a piecewise linear difference-differential equation, we can utilize eigenvalue approach to analyze its dynamics in each linear region separately. There are total three equilibria: $x_0 = 0$ and $x_{-1} = -20/23 \approx -0.87$, $x_1 = 20/23 \approx 0.87$, when the system is without delay, i.e., $\tau = 0$, x_0 is unstable while x_{-1} and x_1 are stable. From [6] we know that for a first-order linear difference-differential equation with constant coefficients

$$\frac{dx}{dt} = ax(t) + bx(t - \tau) \tag{4}$$

the origin is asymptotically stable for any $\tau \in [0, \infty)$ iff

$$a + b < 0 \tag{5}$$

and

$$b - a > 0 \tag{6}$$

clearly for (3), none of the equilibria x_{-1}, x_0, x_1 satisfies conditions (5) and (6) simultaneously. For x_0, (5) does not satisfy, so for any $\tau \in [0, \infty)$, x_0 is unstable. For x_{-1} and x_1 (5) holds but (6) does not hold, so although when $\tau = 0$, x_{-1} and x_1 are asymptotically stable, as τ increases, x_{-1} and x_1 may become unstable and the system may be through bifurcations to chaos. Our computer simulation confirms this and from Figs. 2 and 3, we can see that the solution oscillates around x_1. We also found in our simulation that when the initial condition is chosen as any positive constant function on [−25,0], then the solution oscillates around x_1. Inversely, if the initial condition is chosen as any negative constant function on [−25, 0], then solutions oscillate around x_{-1}. This is because the property of the odd symmetry of (1). Furthermore, in our all trials with various function as the initial condition only one of the two cases mentioned above occurs, no other cases (for example, double-scroll-like) were found. This indicates that the two kind of pseudoattractors (around x_{-1} or x_1) are relatively more "stable" than chaos in higher dimension systems. Thus, the basins of attraction of the two pseudoattractors are roughly ($-\infty$, 0) and (0, $+\infty$), respectively. Detailed bifurcation phenomena will be discussed in forthcoming paper.

Fig. 4 is the power spectrum of $x(t)$, 2^{14} Runge–Kutta iterations were performed with a step size equal to 0.5. The figure shows the first 2^{12} spectral components with normalized frequencies in log-log scale.

IV. Conclusion

A first-order piecewise linear continuous-time system with delay was constructed, which can exhibit chaotic behavior. A pseudoattractor was displayed, the dynamics of the system were briefly analyzed. The present example shows that chaos can occur in one-dimensional continuous-time systems with delay.

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Elementary Canonical State Models of Chua's Circuit Family

Jiří Pospíšil and Jaromír Brzobohatý

Abstract— **Two simple state models of the third-order autonomous piecewise-linear (PWL) dynamical system, topologically conjugate to Chua's circuit family, are proposed. Unlike the known canonical systems they are canonical also with respect to the relation between their parameters and the corresponding eigenvalues, i.e., their state equations contain minimum nonzero coefficients directly determined by the equivalent eigenvalue parameters associated with the two regions of PWL vector field in $\mathbf{R}^3$. The corresponding circuit models containing integrators and adders are introduced. The mutual linear conjugacy between these two elementary canonical state models and their relation with Chua's circuit family is suggested. The computer generated chaos for the double-scroll attractor using one of the developed models is shown.**

I. Introduction

Third-order piecewise-linear (PWL) dynamical systems are intensively studied as the simplest autonomous systems which can exhibit chaotic behavior [1]–[4]. Such systems belong to the so called Class C of vector fields in $\mathbf{R}^3$ [4] and can be described by the general matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}h(w) \tag{1}$$

where $w = \boldsymbol{\alpha}^T\mathbf{x}$, $\mathbf{A} \in \mathbf{R}^{3\times3}$, $\mathbf{b} \in \mathbf{R}^3$, $\alpha \in \mathbf{R}^3$. The simple memoryless PWL function

$$h(w) = \tfrac{1}{2}(|w + 1| - |w - 1|) \tag{2}$$

is continuous and odd-symmetric partitioning $\mathbf{R}^3$ by two parallel planes $U_1 : w = 1$ and $U_{-1} : w = -1$ into inner region $D_0(-1 \leq w \leq 1)$ and two outer regions $D_{+1}(w > 1)$ and $D_{-1}(w < -1)$ as shown in Fig. 1.

The dynamical behavior of such systems is given by two sets of three eigenvalues determining two characteristic polynomials associated with the corresponding regions [3], [4], i.e.,

$$\begin{aligned} D_0\colon \det(s\mathbf{1} - \mathbf{A}_0) &= (s - \mu_1)(s - \mu_2)(s - \mu_3) \\ &= s^3 - p_1 s^2 + p_2 s - p_3 \end{aligned} \tag{3}$$

$$\begin{aligned} D_{+1},\, D_{-1}\colon \det(s\mathbf{1} - \mathbf{A}) &= (s - \nu_1)(s - \nu_2)(s - \nu_3) \\ &= s^3 - q_1 s^2 + q_2 s - q_3 \end{aligned} \tag{4}$$

where

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{b}\boldsymbol{\alpha}^T \tag{5}$$

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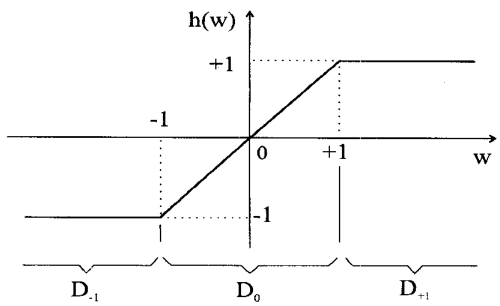

Fig. 1. Basic memoryless PWL feedback function.

and **1** is the unity matrix. State matrix **A** determines coefficients (equivalent eigenvalue parameters) q_1, q_2, q_3 while matrix $\mathbf{A}_0$, i.e., scalar product $\mathbf{b}\boldsymbol{\alpha}^T$, determines coefficients p_1, p_2, p_3. Some of the known systems topologically conjugate to Class *C* (e.g., Chua's universal circuit [3] or Chua's oscillator [4]) are canonical with respect to the behavior, i.e., capable of realizing all possible behavior of the associated vector field, and with respect to the number of circuit elements, i.e., containing the minimum number of elements necessary [7]. They are not canonical, however, with respect to the relation between their parameters and the equivalent eigenvalue parameters [3], [4]. Such a property is useful for both the theoretical and practical studies of chaos. In the present letter two new canonical forms of the state equations, in which this relation is quite elementary, are introduced. The integrator-based circuit models, their relations with Chua's circuit family and use in chaos modeling are shown.

II. Elementary Canonical State Equations

Lemma: Any third-order autonomous PWL dynamical system of Class *C* is elementary canonical, i.e., with respect to the behavior, to the number of free parameters, and to the relation between these parameters and the corresponding equivalent eigenvalue parameters, if state matrix **A** and vectors **b**, $\boldsymbol{\alpha}$ in (1) and (2) have one of the following dual forms:

1st form:

$$\mathbf{A} = \begin{bmatrix} q_1 & -1 & 0 \\ q_2 & 0 & -1 \\ q_3 & 0 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} \quad \boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

2nd form:

$$\mathbf{A} = \begin{bmatrix} q_1 & q_2 & q_3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\alpha} = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix}. \tag{7}$$

Proof: Both the forms of state matrix **A** in (6) and (7) evidently determine the characteristic polynomial (4) because they represent the known canonical forms of the linear system state matrix [5], [6]. They have the minimum number of nonzero parameters, three of them being directly given by the equivalent eigenvalue parameters q_1, q_2, q_3. Substituting **A**, **b**, $\boldsymbol{\alpha}$ from (6) and (7) into (5) the two related forms of matrix $\mathbf{A}_0$ are obtained:

1st form:

$$\mathbf{A}_0 = \begin{bmatrix} q_1 & -1 & 0 \\ q_2 & 0 & -1 \\ q_3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} [1\,0\,0] = \begin{bmatrix} p_1 & -1 & 0 \\ p_2 & 0 & -1 \\ p_3 & 0 & 0 \end{bmatrix} \tag{8}$$

2nd form:

$$\mathbf{A}_0 = \begin{bmatrix} q_1 & q_2 & q_3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [p_1 - q_1 \; p_2 - q_2 \; p_3 - q_3] = \begin{bmatrix} p_1 & p_2 & p_3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \tag{9}$$

These matrices determine the characteristic polynomial (3) because they have formally the same forms as matrix **A** in (6) and (7).

Then the complete forms of elementary canonical state equations, which represent the simplest description of autonomous PWL dynamical third-order systems, are the following:

1st form:

$$\begin{aligned} \dot{x}_1 &= q_1 x_1 - x_2 + (p_1 - q_1)h(x_1) \\ \dot{x}_2 &= q_2 x_1 - x_3 + (p_2 - q_2)h(x_1) \\ \dot{x}_3 &= q_3 x_1 + (p_3 - q_3)h(x_1) \end{aligned} \tag{10}$$

2nd form:

$$\begin{aligned} \dot{x}_1 &= q_1 x_1 + q_2 x_2 + q_3 x_3 + h(w) \\ \dot{x}_2 &= -x_1 \\ \dot{x}_3 &= -x_2 \\ w &= (p_1 - q_1)x_1 + (p_2 - q_2)x_2 + (p_3 - q_3)x_3. \end{aligned} \tag{11}$$

III. Integrator-Based Circuit Models

The direct modeling of the first elementary canonical form (10) leads to the block diagram consisting of three ideal noninverting integrators, three adders, and one PWL element [Fig. 2(a)]. This structure is based on the first canonical state model of the third-order nonautonomous linear system (Analogue Computer network [5]) completed by a nonlinear feedback block determined by a simple memoryless PWL function (Fig. 1) where $w = x_1$. The equivalent eigenvalue parameters q_1, q_2, q_3 and the differences $p_1 - q_1$, $p_2 - q_2$, $p_3 - q_3$ determine the individual adder gains. This structure can easily be modified by using one additional adder and then both the sets

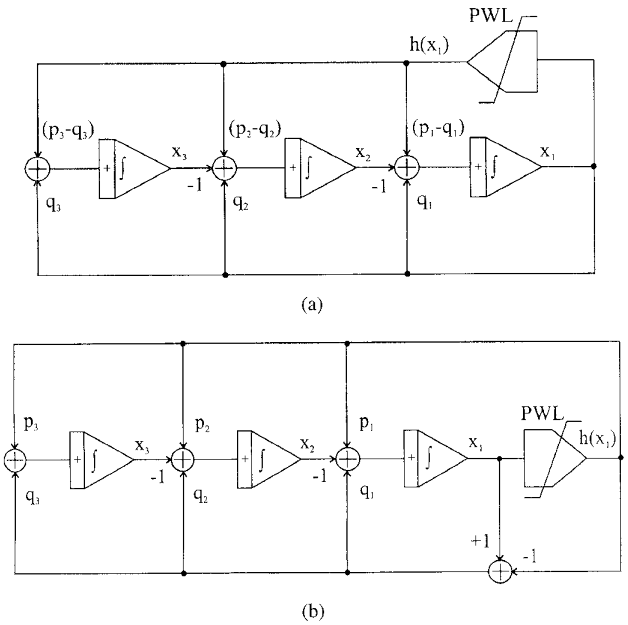

Fig. 2. First elementary canonical state models of the third-order autonomous piecewise-linear dynamical system. (a) Three adder- and (b) four adder-structure.

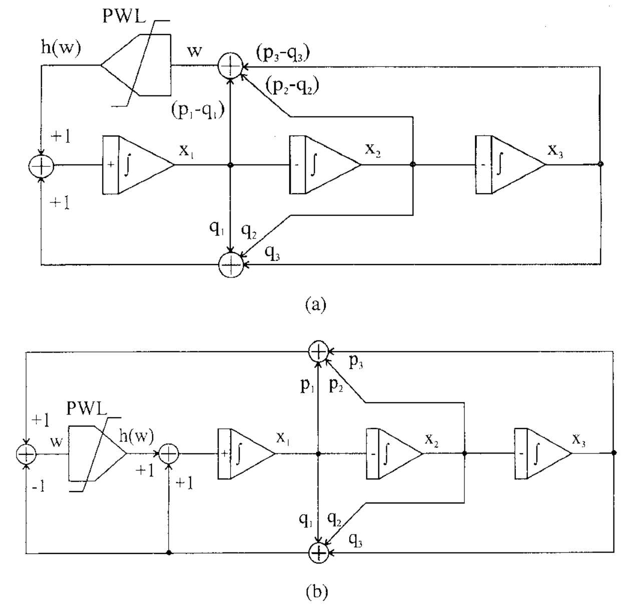

Fig. 3. Second elementary canonical state models of the third-order autonomous piecewise-linear dynamical system. (a) Three adder- and (b) four adder-structure.

of parameters, i.e., p_1, p_2, p_3 and q_1, q_2, q_3, directly determine the corresponding adder gains [Fig. 2(b)]. Unlike Chua's circuit family, PWL function $h(x_1)$ appears in all three state equations but in the circuit models it is represented by one network element only (Fig. 2).

Starting from (11) the complementary block diagram having the same number of network elements (two of the integrators are here inverting) is obtained as shown in Fig. 3(a). This structure is based on the second canonical state model of the third-order linear system (Follow-the-Leader-Feedback network [6]) again completed by a nonlinear feedback determined by the simple PWL function (2) where $w = (p_1 - q_1)x_1 + (p_2 - q_2)x_2 + (p_3 - q_3)x_3$. Utilizing the same procedure as in the previous case, the modified structure having both the sets of equivalent eigenvalue parameters directly determined by the adder gains is found [Fig. 3(b)].

IV. Linear Topological Conjugacy

As both the proposed models as well as Chua's circuit family are qualitatively equivalent in their dynamical behavior, all their mutual relations can generally be expressed by a linear topological conjugacy condition. Its simplified form derived in [4] for Chua's circuit family is valid only for the special case when $\boldsymbol{\alpha}^T = [1 \quad 0 \quad 0]$, i.e., $w = x_1$. As follows from (6) it corresponds also to the first elementary canonical form which is therefore more suitable for direct expressing the linear topological conjugacy with Chua's circuit family in terms of [4].

For the second elementary canonical form (11), where the argument of PWL function $h(w)$ is given by the linear combination of variables x_1, x_2, x_3, the linear topological conjugacy condition must be derived in a generalized form [10]. Then the linear topological conjugacy, where the first form of the elementary canonical state model is given by the second one, can be written in the same explicit form

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}} \tag{12}$$

where $\mathbf{x}$ and $\tilde{\mathbf{x}}$ represent the corresponding state variables [4]. However, the transformation matrix $\mathbf{T}$ must be expressed from the generalized formula [10] which leads to the symmetric form

$$\mathbf{T} = \begin{bmatrix} p_1 - q_1 & p_2 - q_2 & p_3 - q_3 \\ p_2 - q_2 & q_1p_2 - q_2p_1 + p_3 - q_3 & q_1p_3 - q_3p_1 \\ p_3 - q_3 & q_1p_3 - q_3p_1 & q_2p_3 - q_3p_2 \end{bmatrix} \tag{13}$$

as a consequence of the mutual duality of both these state models.

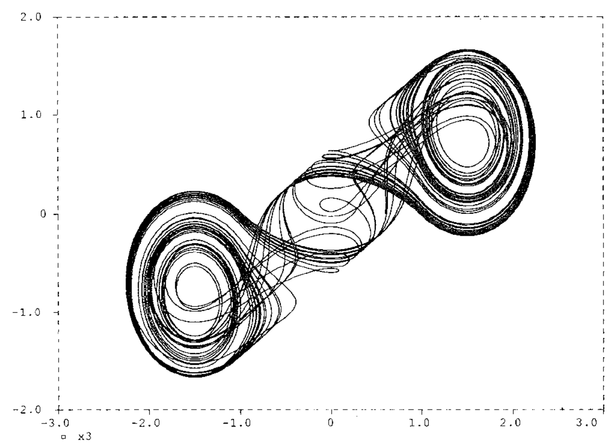

Fig. 4. Computer generated phase portrait in $x_1 - x_3$ plane of the double-scroll attractor represented by the first canonical model with related equivalent eigenvalue parameters [3]: $p_1 = 0.09$; $p_2 = 0.432\,961$; $p_3 = 0.653\,325$; $q_1 = -1.168$; $q_2 = 0.846\,341$; $q_3 = -1.2948$.

V. Example

The properties of the new elementary canonical state models have been numerically verified by PSPICE for all the known sets of the equivalent eigenvalue parameters producing chaotic behavior [3]. As an example, the double-scroll attractor modeled by the first canonical structure of Fig. 2(b) is shown in Fig. 4.

VI. Conclusion

The two proposed state models of the third-order autonomous PWL dynamical system are canonical not only in the sense of the necessary number of free parameters needed for the system design but even in their elementary relation to the corresponding equivalent eigenvalue parameters. Both the models are topologically conjugate to Class C of vector fields in $\mathbf{R}^3$, i.e., qualitatively equivalent to Chua's circuit family, and they are suitable namely for basic studies of the chaotic behavior of PWL systems.

Instead of the simple memoryless feedback PWL function (2) some other types of nonlinear functions can be used (e.g., sigmoid [12], hysteresis [13], etc.) for the modeling and PC simulation of any third-order PWL dynamical system behavior including hyperchaos. Both the models have been extended also for the n-dimensional systems [10] so that it is possible to utilize them even in the study of higher-order chaotic phenomena [7], [11].

All these models can also be used as prototypes for the practical circuit realizations utilizing multiple-input voltage integrators [8] for the first canonical form or multiple-output current integrators [14] for the second canonical form. The digital setting of their parameters and the consequent PC optimization of the resultant network structure are also realizable [9]. Some other theoretical and practical details can be found in [10].

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Comment on "How to Identify Unstable DC Operating Points"

Michael M. Green

Abstract— **The modeling of capacitors and inductors in nonlinear circuits is discussed. In a previous paper by the author a theorem stated that all dependent current (voltage) sources must be terminated by a postive-valued capacitor (inductor). In this letter it is shown that this requirement can be relaxed so that only ports that contain controlling signals are required to be terminated by reactive elements.**

I. Background

In the above paper[1] a number of results regarding the stability of nonlinear circuits were given. The main result of this paper (Theorem 1) was a criterion that can identify the stability of a circuit's operating point that is based only on the circuit's dc elements; this criterion is independent of values and locations of any capacitor or inductor that might be in the circuit. In particular, this criterion states that if a certain constant Γ, derived from a dc circuit linearized at a certain operating point, is negative, then the operating point is unstable. The proof of this result comes from showing that as long as enough capacitors and inductors are appropriately modeled, the linearized circuit's characteristic equation can always be normalized such that the highest-order coefficient must be positive and the constant term is Γ, which does not depend on any capacitor or inductor values.

How capacitors and inductors are appropriately modeled was addressed in another result, which was derived as a byproduct of the development of the stability criterion. In particular, Theorem 2 of the above paper states that any dependent current (voltage) source must have a capacitor (inductor) placed in parallel (series) with it. This requirement holds even if the value of a dependent source gain is zero. (Such zero-valued dependent sources are needed when a port corresponds to an open-circuit voltage or a short-circuit current that is used as a controlling signal but is not directly connected to an actual, nonzero dependent source.) This result is important because there are locations in certain circuits where capacitors and inductors *must* be modeled in order to observe unstable natural frequencies. The dangers of leaving out such critical capacitors and inductors are illustrated in Fig. 3(b) of the above paper.

As an example, consider the simple model of an operational amplifier, shown in Fig. 1 with the two reactive elements required by Theorem 2 of the above paper included. In particular, a capacitor C must be modeled across the input terminals; since its voltage is a controlling signal, it must be considered to be a zero-valued dependent current source. Also, an inductor L must be modeled in series with the dependent voltage source at the output. However, the series inductor is seldom considered important when analyzing the dynamics of most op-amp circuits. In the next section we will give a new result that relaxes the topological modeling requirements given in Theorem 2 of the above paper and hence justifies this observation.

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