The Evolution of Spatio-Temporal Disorder in a Chain of Unidirectionally-Coupled Chua's Circuits

Grigory V. Osipov and Vladimir D. Shalfeev

Abstract—**The nonlinear dynamics of a chain of unidirectionally coupled Chua's circuits is investigated. The nonlinearity is chosen to be smooth and the coupling between cells in the chain is linear. The phenomena of oscillatory spectrum complication along the chain caused by developing instabilities, suppression of oscillations, and chaotic synchronization of the cells are demonstrated.**

I. Introduction

INVESTIGATIONS OF the spatio-temporal dynamics of cellular nonlinear networks (CNN) consisting of a large number of coupled active cells [1] generally encounter considerable difficulties. Some progress has been made in the investigation of the dynamics of networks having a regular configuration, in particular, in array models for which one can exploit the spatial homogeneity of the medium [2].

In this paper we analyse some properties of the spatio-temporal dynamics of a one-dimensional (1-D) CNN consisting of coupled Chua's circuits [3] when the coupling between the cells is unidirectional. This model is interesting not only as a particular case of a 1-D CNN with mutual coupling, but also as an example of a *flow* system.

The dynamics of *flow* systems modelled by chains of unidirectionally coupled active cells had been actively studied in the literature [2], [4]–[9], where Van der Pol generators, rotators, and inertial generators were chosen as the cells. Spatial bifurcations giving rise to chaos had been observed. The regime was periodic at the beginning of the chain. But as the generator number increases, the periodic regime bifurcates into a quasi-periodic regime which, in turn, bifurcates into a *chaotic* regime. It was shown that using this approach one can explain the spatial evolution of turbulence following the universal Feigenbaum's law. The authors of [2], [7] considered as an example a chain of unidirectionally-coupled rotators in which the Landau–Hopf scenario of complex dynamics is realized, i.e., the more rotators interact with each other, the more complicated are the motions due to the appearance of new frequencies incommensurate with the previous ones, thereby increasing the dimension of the resulting quasi-periodic motions.

In this paper we choose Chua's circuits as the cells of a 1-D CNN. It is known [10], [11] that an extremely complicated dynamics is inherent in the individual Chua's circuit cells

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The authors are with Nizhny Novgorod University, Nizhny Novgorod 603600, Russia.

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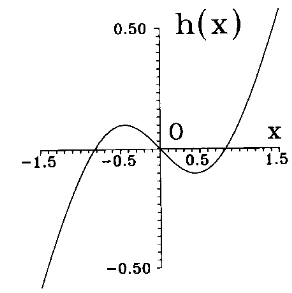

Fig. 1. Smooth nonlinearity defined by (4) is chosen for Chua's circuits in this paper.

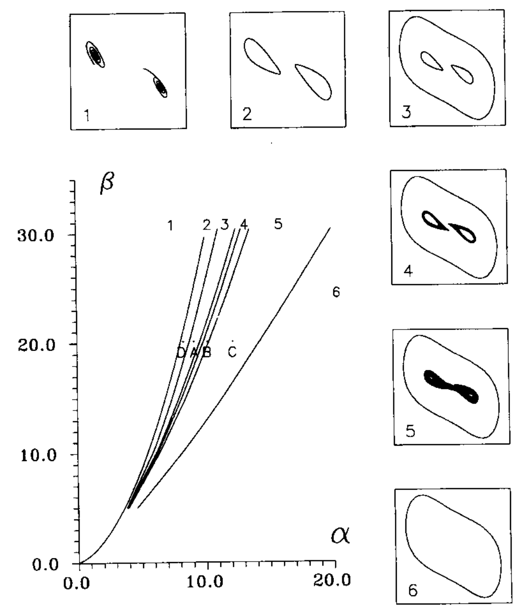

Fig. 2. Bifurcation plane of the parameters for system (3).

even when couplings are not taken into account. Therefore it is to be expected that not only complex regimes typical of the nonlinear partial dynamics of Chua's circuits, but also other spatio-temporal phenomena stimulated by the collective interaction of cells may emerge in the coupled CNN [3], [12]–[16]. It is not our goal to give a more or less complete description of the spatio-temporal dynamics of 1-D CNN consisting of Chua's circuits. Instead, we will consider only some phenomena which appear to be important for both theory and applications, namely, the instability development and the suppression of oscillations along the chain and chaotic synchronization. These spatio-temporal phenomena are evidently not only typical of the 1-D CNN of Chua's circuits considered here but have a more general nature as well [2].

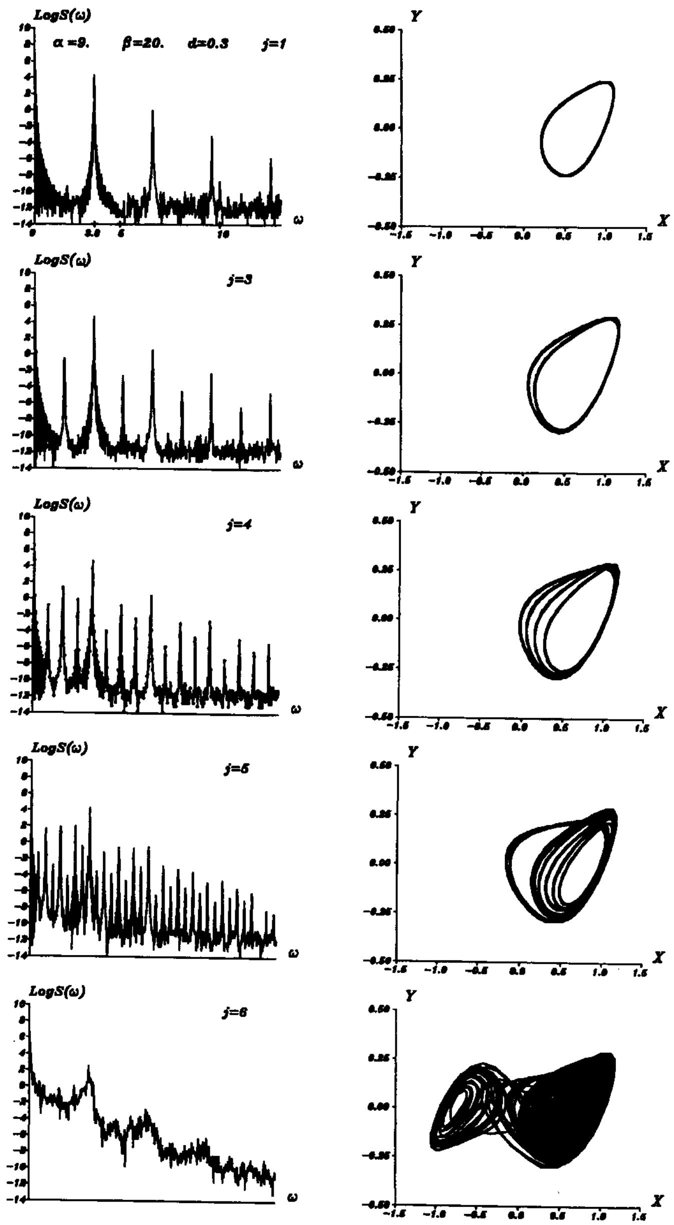

Fig. 3. Evolution of periodic motion in a chain and the transition to chaos.

II. A Model of 1-D CNN of Chua's Circuits

We consider a 1-D CNN consisting of unidirectionally coupled Chua's circuits. In the absence of coupling, a mathematical model of a single Chua's circuit may be written in the form [10]

$$\begin{aligned}\dot{x} &= \alpha(y - x - f(x))\\ \dot{y} &= x - y + z\\ \dot{z} &= -\beta y.\end{aligned} \tag{1}$$

The nonlinearity $f(x)$ is usually approximated by a three- or five-segment piece-wise-linear function. In modeling the dynamics we tried to have in the mathematical model a minimal number of parameters. For this purpose we consider a smooth nonlinearity of the form

$$f(x) = cx - \frac{2c_0 x}{1 + (c_0 x)^2}. \tag{2}$$

Then the system (1) may be written as

$$\begin{aligned}\dot{x} &= \alpha(y - h(x))\\ \dot{y} &= x - y + z\\ \dot{z} &= -\beta y,\end{aligned} \tag{3}$$

where

$$h(x) = c_1 x - \frac{2c_0 x}{1 + (c_0 x)^2}, \qquad c_1 = 1 + c. \tag{4}$$

Modeling the system (3) with nonlinearity (4) showed that the partitioning of the plane of the parameters (α, β) into domains of the existence of different regimes is analogous to that available in the literature for a piecewise-linear function $f(x)$. Modelling was made for the nonlinearity (4) at $c_0 = 0.7$ and $c_1 = 1.05$. The form of the function $h(x)$ is given for this case in Fig. 1. Fig. 2 presents some bifurcation curves on the plane (α, β) (they will be needed for our further analysis).

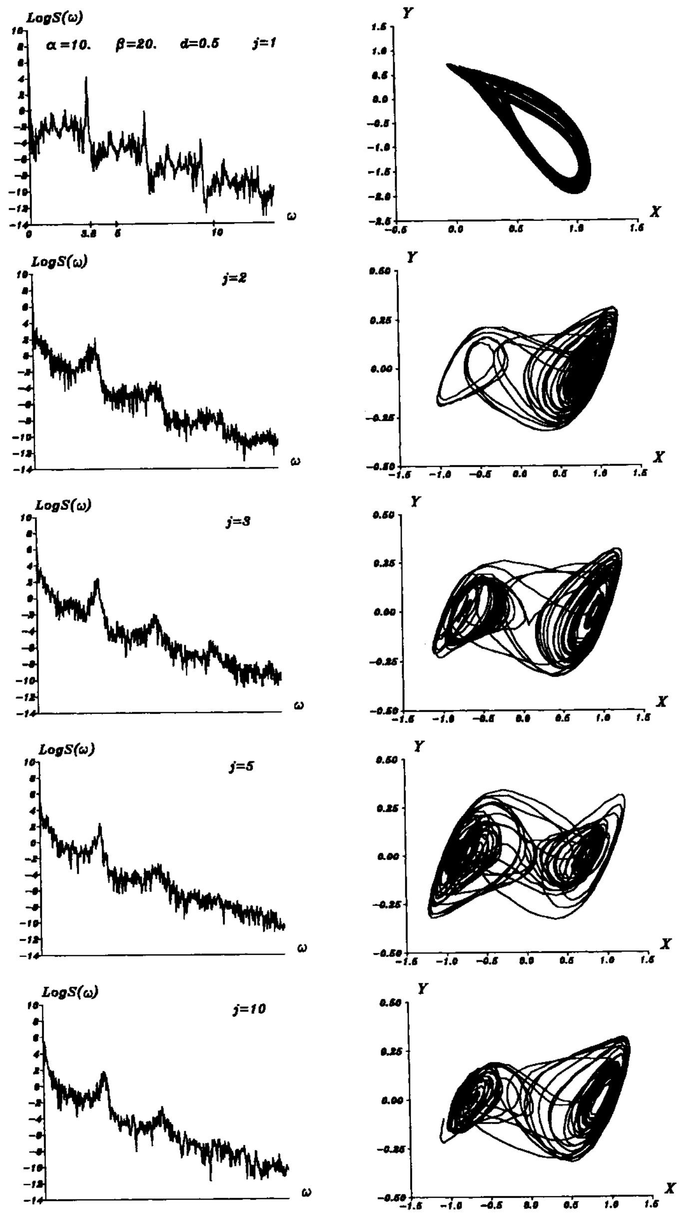

Fig. 4. Phenomenon of chaotic synchronization.

We now consider a model of a 1-D CNN. Obviously, the CNN dynamics depends on the type and the value of *coupling* between the cells, although it is difficult to give *a priori* preference to one or another type of coupling because the problem is essentially nonlinear. For simplicity, we restrict ourselves in this paper to the case when the neighboring cells are coupled linearly and only in one coordinates direction (increasing cell number). Hence, the model of this 1-D CNN is described by:

$$\begin{aligned}\dot{x}_j &= \alpha(y_j - h(x_j)) + dx_{j-1}\\ \dot{y}_j &= x_j - y_j + z_j\\ \dot{z}_j &= -\beta y_j, \qquad j = 1, 2, \cdots, N.\end{aligned} \tag{5}$$

We assume that the condition $x_0(t) = 0$ is fulfilled at the boundary. An important assumption made in the model (5) is that all cells in the chain as well as couplings are identical, which means that the chain is spatially homogeneous. The number of the cell j is regarded as the *spatial* coördinate. It is also worthy of notice that for the coupling considered it is natural to regard the coupling coefficient d as a control parameter since it can give rise to different oscillatory regimes in the CNN cells. This is connected with the fact that the desired regime of oscillation in a single CNN cell may be chosen in accordance with the partitioning of the plane of the parameters (α, β), and the coupling signal may be considered as a known external forcing of each subsequent cell by the preceding one.

Let us now chose our initial regime in the *first* cell of the chain (5) to be a regime of regular periodic oscillations and then consider the scenarios of the evolution of such a periodic regime with the increase of the number of the cell, i.e., along the chain.

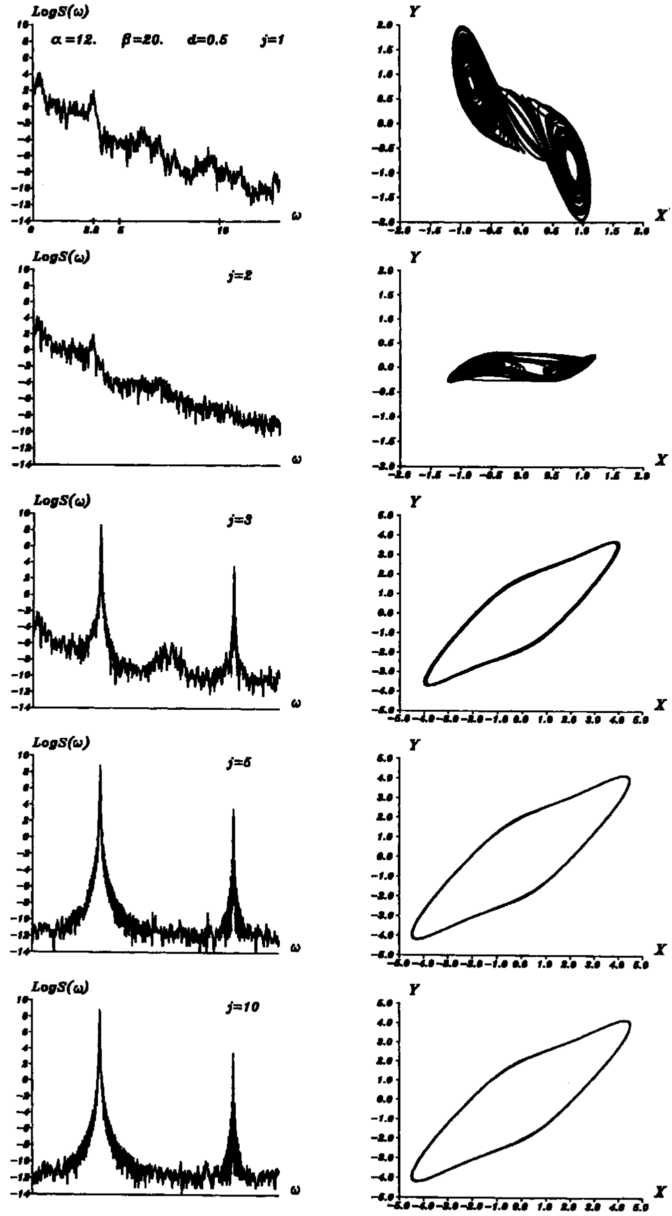

Fig. 5. Suppression of chaotic oscillations along the chain.

III. The Development of Instabilities and Spatial Transition to Chaos

For the values of the parameters $\alpha = 9, \beta = 20, d = 0.3$ (see Fig. 2, point A) in (5), we found that, as the spatial coordinate j is increased, the periodic oscillations change in accordance with the following scenario. Owing to developing instabilities, the periodic oscillations in the first cell transform through a series of spatial period-doubling bifurcations to chaotic oscillations. The spectra and projections of the attractors onto the (x_j, y_j)-plane are given in Fig. 3 for five cells of the chain ($j = 1, 3, 4, 5, 6$). Periodic mutually synchronized oscillations (period T_0) occur at frequency $\omega_0 \approx 3.8$ in the first and second cells. The oscillatory regime of a doubled period $2T_0$ is realized in the third cell, oscillations of period $4T_0$ occur in the fourth cell, and of period $8T_0$ in the fifth cell. The regime of chaotic oscillations that looks like a double-scroll Chua's attractor in space (x_6, y_6, z_6) emerges in the sixth cell.

Note that, depending on the values of the parameters of the individual cells, and of the coupling d, not all of the sequence of period doubling motions may be realized in the chain. This is explained by the discreteness of the spatial coordinate j which is the control parameter in the transition to chaos. Consequently, the excitation of chaotic oscillations (such as a double-scroll or a spiral Chua's-attractor) may occur only for certain values of d.

IV. Spatial Chaotic Synchronization

If we choose the parameters of the chain (5) such that each individual cell oscillates chaotically, then a regime of spatial chaotic synchronization is established in the CNN for definite values of the coupling coefficient d. The result of our experiments on the model (5) are presented in Fig. 4 for $\alpha = 10$ and $\beta = 20$ (point B in Fig. 2). The regime of chaotic oscillations sets in any individual uncoupled cell depending

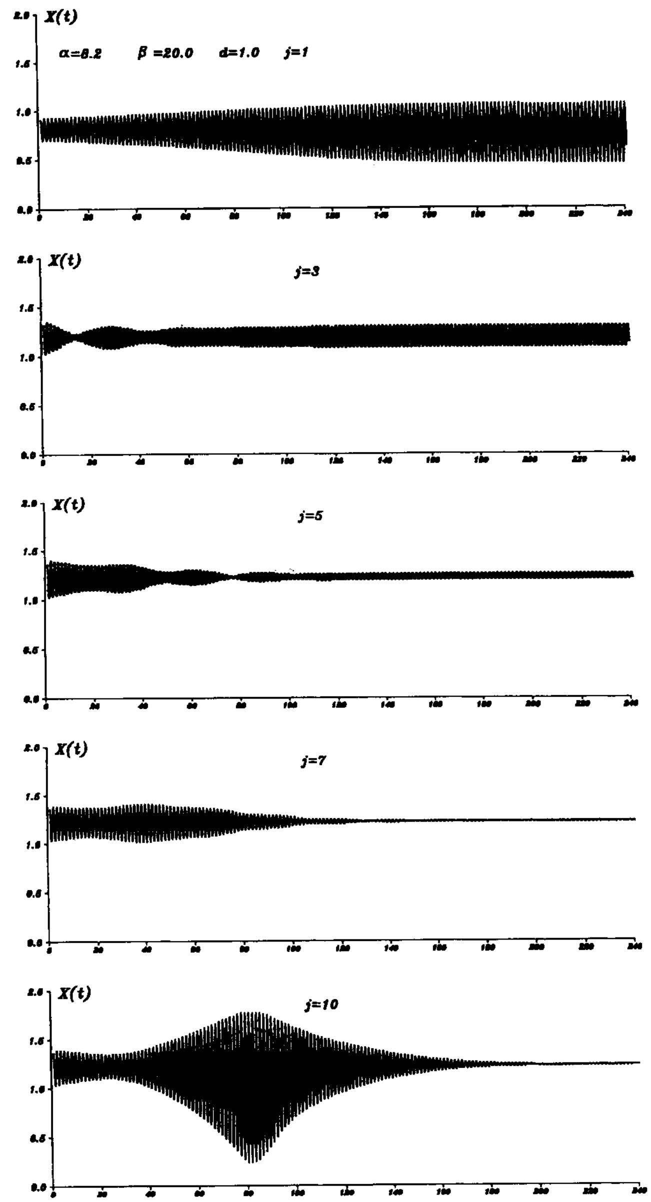

Fig. 6. Suppression of periodic oscillations along the chain.

on initial conditions. Therefore, a spiral Chua's-attractor is observed in the first cell in Fig. 4. If we introduce the coupling $d > 0$, then, as j is increased, the chaos is transformed and an attractor of the double-scroll type is realized in the second cell. It is nonsymmetric because the forcing of the second cell by the first one consists of an oscillation in the positive half-space for the variable x_1 (i.e., the average value of x_1 is positive). With a further increase of j, double-scroll attractors occur in the third and subsequent cells as well. The oscillation intensity remains almost unchanged and the spectrum becomes smoother. A regime of chaotic *synchronization* is observed for $j \geq 5$ where no pronounced changes appear in the spectra of oscillations, as is evident in Fig. 4 for $j \geq 5$.

V. Suppression of Chaotic and Periodic Oscillations

Since the domain of the existence of chaotic oscillations on the $\alpha - \beta$ parameter plane borders on regions of regular motions it should be expected that the change of the coupling parameter d may, generally speaking, lead to a loss of chaoticity, i.e., a cell which has had chaotic partial dynamics is transformed into a regime of periodic regular oscillations when coupling is introduced. Fig. 5 demonstrates the suppression of chaotic oscillations along the chain (here $\alpha = 12, \beta = 20$ and $d = 0.5$, point C in Fig. 2). The chaotic oscillations which are rather intense in the first few elements are gradually suppressed, as j is increased, and we can observe that a regime of periodic oscillations is already realized for $j \geq 5$. Thus, clusters of cells oscillating regularly or chaotically are formed in the chain. A remarkable result we obtained here is that the width of the regular and chaotic clusters (i.e., the number of cells) may be controlled.

Not only chaotic but also regular periodic regimes may be suppressed along the chain [2]. As an example, we show in Fig. 6 the change, as j increases, turn a periodic motion regime to a static regime at $\alpha = 8.2, \beta = 20$ and $d = 1.0$

(point D in Fig. 2). This is the spatial analog of the inverse Andronov–Hopf bifurcation phenomena for increasing values of j.

VI. Conclusion

For a unidirectionally coupled chain of Chua's circuits with increasing j, we have demonstrated the following phenomena: spatial development of excitations and the transition to chaos, spatial chaotic synchronization, and spatial suppression of excitations. Of course, these phenomena of spatio-temporal dynamics do not give a complete picture of the dynamic behavior of the chain of interest. However, they are rather typical of the dynamics of the chains of coupled Chua's circuits. It can be expected that these phenomena will occur in more general cases, in particular, in more complicated types of unidirectional couplings (nonlinear coupling, coupling along several coordinates, etc.), in 2-D CNN, in CNN consisting of generalized Chua's circuits, etc.

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Grigory V. Osipov graduated from the Faculty of Applied Mathematics and Cybernetics of Nizhny Novgorod State University, Russia, in 1982 and received the Ph.D. degree in 1989 in physics and mathematics.

Currently, he is an Associate Professor at Nizhny Novgorod University. His research concerns the theory of discrete nonlinear media. He has written over 30 scientific papers and 1 book, *Stability, Structures and Chaos in Nonlinear Synchronization Networks*, Singapore: World Scientific, 1995.

Vladimir D. Shalfeev received the Diploma in 1963 and the Ph.D. degree in 1969, both in physics and mathematics from the University of Nizhny Novgorod, Russia. He received the Doctor of Science degree in physics and mathematics in 1990.

He became a Professor in 1991. Currently, he is a Professor and Chairman of the Department of Theory of Oscillations at Nizhny Novgorod University. His principal interests are nonlinear dynamics, spatio–temporal chaos with applications in signal processing and communication, synchronization theory, and dynamics of phase-locked loops. He has written over 100 scientific papers and 2 books, among them *Stability, Structures and Chaos in Nonlinear Synchronization Networks*, Singapore: World Scientific, 1995.

Dr. Shalfeev is a corresponding member of the Academy of Engineering Sciences of Russia.